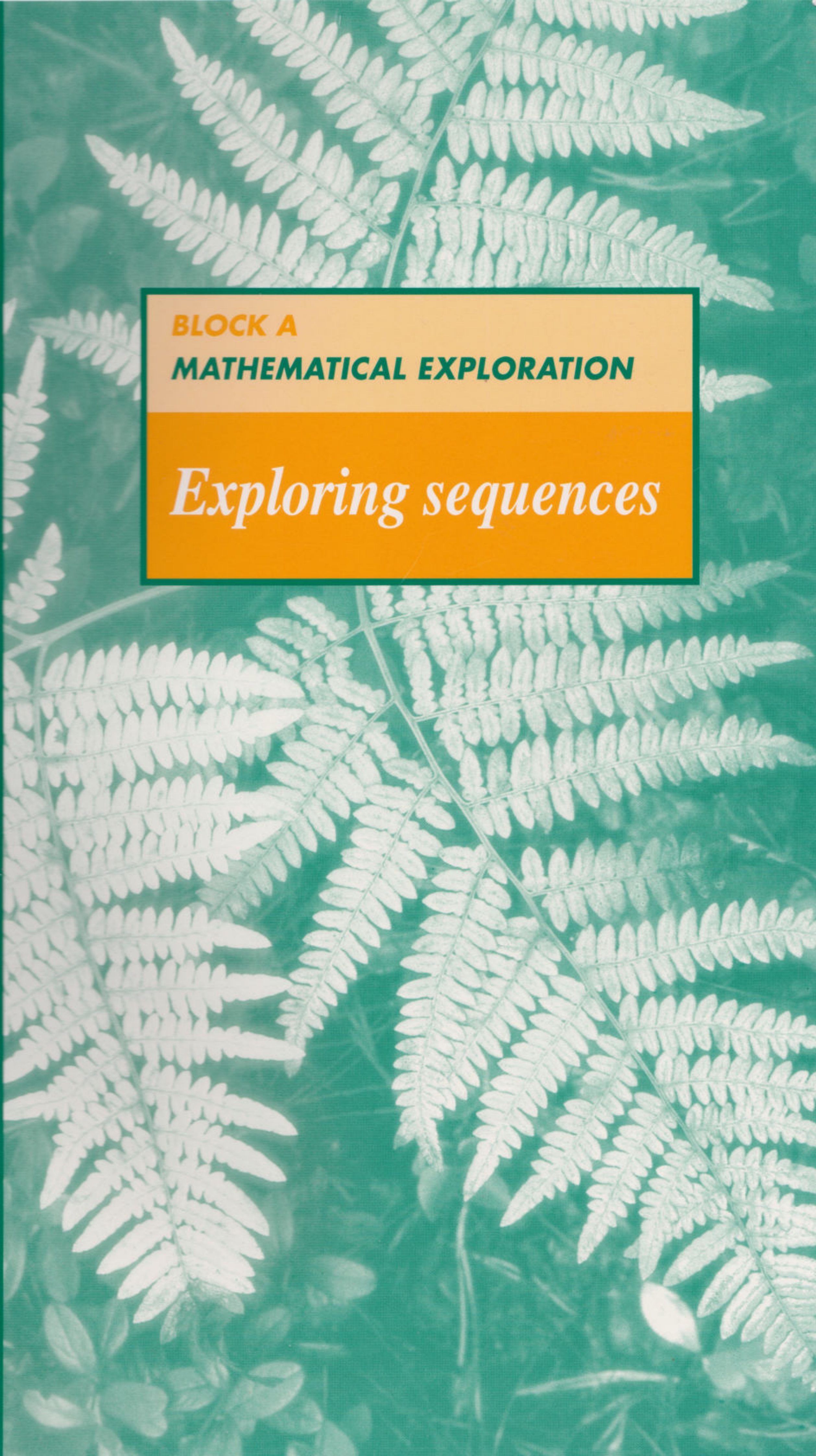
MS221 Chapter A1

The Open University

A second level interdisciplinary course

Exploring Mathematics



BLOCK A

MATHEMATICAL EXPLORATION

# *Exploring sequences*

CHAPTER

A1

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A second level interdisciplinary course

Exploring Mathematics

**BLOCK A**

**MATHEMATICAL EXPLORATION**

# Exploring sequences

*Prepared by the course team*

**CHAPTER A1**

## About this course

This course, MS221 *Exploring Mathematics*, and the courses MU120 *Open Mathematics* and MST121 *Using Mathematics* provide a flexible means of entry to university-level mathematics. Further details may be obtained from the address below.

MS221 uses the software program Mathcad (MathSoft, Inc.) to investigate mathematical concepts and as a tool in problem solving. This software is provided as part of the course.

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# *Introduction to Block A*

The idea of exploring mathematics may seem rather strange. Mathematics is often thought to be a subject in which there are no uncertainties and nothing is left to be discovered. It is true that much of mathematics is simply a tool for solving numerical and spatial problems. But even such 'useful' mathematics took centuries to reach its present form, developing step by step thanks to the ideas of many individuals, and mathematics continues to develop as new challenges require new techniques or new ways of using old techniques.

This course introduces you to a number of important topics in mathematics and also aims to give you some feeling for how they developed. Topics in mathematics may develop not just because they are needed for applications but because individuals who find the topics attractive in themselves become interested in exploring and understanding them, and thereby discover challenging problems and elegant solutions. In MS221 there will be less emphasis on the applications of mathematics than in MST121, and more emphasis on the mathematics itself and how it developed. Applications will certainly not be ignored, but they will play a less prominent role.

Block A introduces several aspects of mathematical exploration. Chapter A1 begins with two rather old mathematical problems, one about rectangles and another about breeding rabbits. These lead to a number, called the *golden ratio*, and to a sequence, the Fibonacci numbers, which turn out to be closely related to each other. Exploring these problems involves several basic mathematical activities, including noticing patterns in a sequence and using algebra to show that these patterns arise from general properties of the sequence.

Chapter A2 is about the curves known as *conics*, which were first studied by the Ancient Greeks as cross-sections of cones and which became important centuries later in science. We study conics using algebra and find that they have elegant and unexpected properties, which give them applications in diverse areas.

Chapter A3 explores the idea of a *function*, extending it beyond the basic idea of a function given in MST121, Chapter A3. In that chapter, most functions consisted of a rule relating a dependent and an independent real variable, but here you will see the advantage of using functions arising from geometric situations, which relate more general variables. In particular, such functions are used to help understand conics and to derive trigonometric properties.

# Study guide

There are five sections to this chapter. They are intended to be studied consecutively in five study sessions. Each section will require two to three hours to study.

All sections require the use of this main text, Section 3 requires the use of an audio cassette player, and Section 4 requires the use of the computer together with Computer Book A. Subsections 1.3, 2.2 and 5.3 are 'for interest only' and will not be assessed.

The pattern of study for each session might be as follows.

Study session 1: Section 1.

Study session 2: Section 2.

Study session 3: Section 3.

Study session 4: Section 4.

Study session 5: Section 5.

If it is more convenient for you, Section 5 may be studied before Section 4.


Before studying this chapter, you should be familiar with the following topics:

- ◇ subscript notation for sequences;
- ◇ the idea of a closed form for a recurrence sequence;
- ◇ the formula for solving a quadratic equation;
- ◇ solving pairs of simultaneous equations;
- ◇ techniques for rearranging algebraic expressions;
- ◇ simple geometric properties of triangles;
- ◇ the long-term behaviour of geometric sequences, including the notation '$a_n \to l$ as $n \to \infty$';
- ◇ the formula for the sum of a finite geometric series.

> The optional Video Band A(i) *Algebra workout – Rearranging formulas* could be viewed at any stage during your study of this chapter.

# Introduction

First-order recurrence systems were discussed in MST121, Chapter A1.

This chapter introduces sequences defined by *second-order* recurrence systems, so called because each term of the sequence depends on the *two* preceding terms. It also introduces a number of basic mathematical skills which will be developed in the course. These include:

◇ tackling problems by introducing algebra;

◇ noticing patterns;

◇ proposing general properties, known as conjectures;

◇ constructing proofs using algebraic manipulations.

This topic and these skills are introduced in a particular historical context which has interested mathematicians (and others) for many centuries.

Section 1 introduces a number $\phi$ known as the *golden ratio*, which has played an important role in mathematics since its discovery by the Ancient Greeks. This number is associated with the quadratic equation $x^2 - x - 1 = 0$, whose solutions are:

The Greek letters $\phi$ (phi) and $\psi$ (psi) are often used to denote the solutions of this quadratic equation.

$$\phi = \tfrac{1}{2}(1 + \sqrt{5}) = 1.618\ldots \quad \text{and} \quad \psi = \tfrac{1}{2}(1 - \sqrt{5}) = -0.618\ldots.$$

Here several properties of $\phi$ and $\psi$, and some general properties of solutions of quadratic equations are discussed.

Section 2 introduces the sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots,$$

in which each term (after the second) is the sum of its two predecessors. This is called the *Fibonacci sequence*, after the medieval mathematician who first discussed it. A number of patterns seem to be present in the Fibonacci sequence and, as a result, various general properties about the sequence are proposed. Such a proposed property is called a *conjecture* and an example which shows that a conjecture is false is called a *counter-example*. Fibonacci numbers appear often in nature, and one possible explanation for this, based on a connection between Fibonacci numbers and the golden ratio, is discussed.

Fibonacci numbers are also used in computer science as the basis for efficient methods of manipulating data.

Section 3 includes an audio tape which deals with a general type of second-order recurrence system, of which the Fibonacci sequence is a special case. As a result, a closed form for Fibonacci numbers, called *Binet's formula*, is obtained.

In Section 4, the computer is used to explore the values of the recurrence sequences introduced in Section 3. This leads to various conjectures about such sequences.

In Section 5, some of the conjectures about Fibonacci numbers are proved to be true and one is shown to be false. Various techniques of proof are employed, including algebraic manipulation, the sum of a finite geometric series, Binet's formula and production of a counter-example.

# 1 The golden ratio

In this section, we solve a geometrical problem by using algebra. The solution is a number, called the golden ratio, which has been considered special for over 2500 years. This number appears in many unexpected contexts and it continues to be of importance in mathematics, and in applications of mathematics.

## 1.1 What is the golden ratio?

The following problem was known to the Ancient Greeks.

**The Rectangle Problem**
Suppose that a rectangle has a square removed from one end, leaving a rectangle the same shape as the original rectangle. What is the ratio of the lengths of the sides of the original rectangle?

Before solving this problem, a couple of remarks about it are in order.

◇ In geometry, two figures which have the same shape, but not necessarily the same size, are said to be *similar*. In any two similar figures, the lengths of pairs of corresponding sides are in the same ratio. In particular, the ratios of the lengths of the longer and shorter sides for each of two similar rectangles are equal.

◇ The statement of the Rectangle Problem assumes that the problem *has* a solution and, moreover, that there is just *one* solution. When faced with a problem, it is as well to keep in mind the possibility of *more than one* solution or even *no* solutions.

To obtain some feeling for the problem, we explore several particular rectangles. In Figure 1.1, each of the original rectangles has the shorter side of length 1 unit and the longer sides are of length 1.5 and 2. The rectangles remaining after a square (of side 1 unit) is removed are shaded.

Since we are concerned with ratios of lengths, it is sufficient to take the shorter side of the original rectangle to be of length 1 unit. The actual length of this unit does not matter.

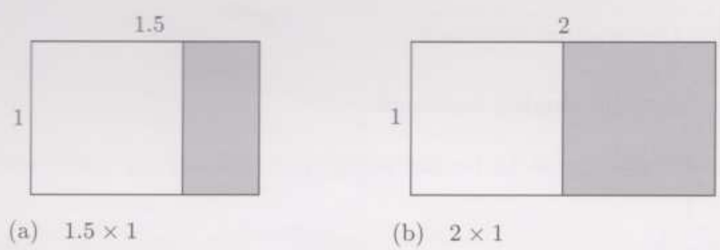


*Figure 1.1* Two particular rectangles

The ratios of the lengths of the longer side to the shorter side for each of the rectangles in Figure 1.1 are recorded in Table 1.1.

*Table 1.1* Ratios of sides of rectangles

| Longer side | Shorter side | Ratio for original rectangle | Ratio for shaded rectangle |
|---|---|---|---|
| 1.5 | 1 | $\frac{1.5}{1} = 1.5$ | $\frac{1}{0.5} = 2$ |
| 2 | 1 | $\frac{2}{1} = 2$ | $\frac{1}{1} = 1$ |

We are seeking an original rectangle for which the corresponding shaded rectangle has the same shape; that is, the ratios of their sides are equal. In Figure 1.1(a) the shaded rectangle is thinner than the original rectangle, whereas in Figure 1.1(b) the shaded rectangle is fatter than the original rectangle. It appears that as the ratio for the original rectangle increases from 1.5 to 2, the corresponding ratio for the shaded rectangle decreases from 2 to 1. This suggests that to solve the Rectangle Problem the ratio must lie between 1.5 and 2. Some further exploration is called for.

### *Activity 1.1 More rectangles*

Calculate the ratios of the lengths of the sides of the remaining rectangle (correct to three decimal places) when the shorter side of the original rectangle is 1 and the longer side takes each of the following values.

(a) 1.6 (b) 1.9 (c) 1.7

What do your calculations suggest about the Rectangle Problem?

Solutions are given on page 39.

To solve the Rectangle Problem, we use algebra. We denote the length of the longer side by the (unknown) variable $x$, and seek an equation satisfied by $x$. The shorter side has length 1, as above, so $x > 1$.

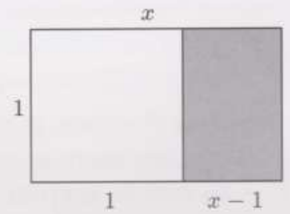


*Figure 1.2* Introducing an unknown

We require that the shaded rectangle is the same shape as the original rectangle; that is, the ratios of their sides are equal. Figure 1.2 shows that the ratio of the longer side to the shorter side is:

$\dfrac{x}{1}$ for the original rectangle;

$\dfrac{1}{x-1}$ for the shaded rectangle.

Since $x > 1$, the expression $x - 1$ has a non-zero value.

As we want these ratios to be the same, the unknown $x$ must satisfy

$$\frac{x}{1} = \frac{1}{x-1},$$

which simplifies to give

$$x(x-1) = 1; \quad \text{that is,} \quad x^2 - x - 1 = 0.$$

This is a quadratic equation with two solutions, given by the quadratic equation formula.

### *Activity 1.2 Solving the quadratic*

(a) Use the quadratic equation formula to find the two solutions of

$$x^2 - x - 1 = 0.$$

(b) Hence solve the Rectangle Problem.

Solutions are given on page 39.

Recall that the solutions of

$$ax^2 + bx + c = 0,$$

where $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

#### *Comment*

The solution to the Rectangle Problem does indeed lie in the range suggested by the particular rectangles considered earlier.

Our discussion of the Rectangle Problem illustrates various features of problem solving in mathematics. In particular, we often:

- ◇ draw pictures to help understand the problem;
- ◇ explore special cases to get a feeling for the solution;
- ◇ introduce notation and express the problem using equations;
- ◇ use algebra to rearrange equations and solve them;
- ◇ check, in some way, that the answer obtained is reasonable.

For a particular problem, you may find that not all these features are appropriate for obtaining a solution.

Here is a similar problem for you to try.

### *Activity 1.3 Another rectangle problem*

> When a certain rectangle is divided in half, the two rectangles obtained are each similar to the original rectangle. What is the ratio of the lengths of the sides of the original rectangle?

(a) Draw a picture to illustrate this rectangle problem.

(b) In the special case when a rectangle with sides of length 1 and 1.5 is divided in half, calculate the ratio of the lengths of the sides of the resulting smaller rectangles. Use your answer to predict whether the solution to this rectangle problem is larger or smaller than 1.5.

(c) Use algebra to solve this rectangle problem.

A solution is given on page 39.

#### *Comment*

The paper sizes A0, A1, A2, A3, A4, A5, ... all have this shape. For example, an A4 sheet (the size of this page) gives two A5 sheets when divided in half.

The original Rectangle Problem is equivalent to the following problem about ratios of lengths along a line.

> Can a line $AB$ be divided by a point $P$ in such a way that the ratio of $AB$ to $AP$ is equal to the ratio of $AP$ to $PB$?


*Figure 1.3* $AB/AP = AP/PB$

The Ancient Greeks called this version of the problem: division of a line into **extreme and mean ratio**. They knew that if a line was divided in this way, then the ratio is $\frac{1}{2}(1+\sqrt{5})$, though they would have expressed this fact in a different way.

Pacioli (1445–1517) was an Italian friar who wrote an important text book on algebra and geometry, and also one on double-entry bookkeeping.

In 1509, Luca Pacioli published an influential book about this ratio, called *Divina proportione* (The divine proportion). This described many occurrences of the ratio in geometric figures, with fine illustrations thought to have been drawn by Pacioli's friend Leonardo da Vinci.

Since the 19th century, the number

$$\phi = \tfrac{1}{2}(1+\sqrt{5}) = 1.618\,033\,988\,7\ldots$$

has been called the **golden ratio**, **golden section** or **golden number**. A rectangle with sides in this ratio is known as a **golden rectangle**.

Later it is shown that $\phi$ is irrational. Some writers prefer to use the Greek letter $\tau$ (tau) for the golden ratio.

We call $x^2 - x - 1 = 0$ the **golden ratio equation**, and denote the other solution of this equation, found in Activity 1.2, by

$$\psi = \tfrac{1}{2}(1-\sqrt{5}) = -0.618\,033\,988\,7\ldots.$$

In other chapters of this course $\phi$ and $\psi$ may be assigned different uses.

From the similarity of the decimals for $\phi$ and $\psi$, it appears likely that $\phi + \psi = 1$. In the next subsection, we see that this is a special case of a general property of quadratic equations.

## 1.2 Properties of solutions of quadratic equations

The fact that $\phi + \psi = 1$ can be confirmed directly as follows:

$$\begin{aligned}\phi + \psi &= \tfrac{1}{2}(1+\sqrt{5}) + \tfrac{1}{2}(1-\sqrt{5})\\ &= \tfrac{1}{2} + \tfrac{1}{2} = 1.\end{aligned}$$

The cancellation here illustrates that it is sometimes useful to work with such numbers in their *exact* forms, rather than evaluating the square roots to give decimals. Here are two further calculations for you to do this way.

### Activity 1.4 Calculations involving $\phi$ and $\psi$

Use the exact forms of $\phi$ and $\psi$ to verify the following properties of $\phi$ and $\psi$.

(a) $\phi\psi = -1$

(b) $\phi - \psi = \sqrt{5}$

Solutions are given on page 40.

Now suppose that we want to express

$$\frac{1}{\phi} = \frac{1}{\frac{1}{2}(1+\sqrt{5})} = \frac{2}{1+\sqrt{5}} \tag{1.1}$$

as a fraction with no square root in the denominator. One method is to use the equation $\phi\psi = -1$. This gives $1/\phi = -\psi$, so

See Activity 1.4(a).

$$\frac{1}{\phi} = -\psi = \tfrac{1}{2}(\sqrt{5}-1).$$

This shows that $1/\phi = 0.618\,033\,988\,7\ldots$. Alternatively, we multiply both numerator and denominator of the right-hand side of equation (1.1) by $1-\sqrt{5}$. An application of the difference of two squares then removes the square root from the denominator:

$(a-b)(a+b) = a^2 - b^2$

$$\begin{aligned}\frac{1}{\phi} = \frac{2}{1+\sqrt{5}} &= \frac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} \\ &= \frac{2(1-\sqrt{5})}{1-5} \\ &= -\tfrac{1}{2}(1-\sqrt{5}) \\ &= \tfrac{1}{2}(\sqrt{5}-1),\end{aligned}$$

as before. More generally, we can remove the square roots in the denominator of any expression of the form

$$\frac{1}{\sqrt{p}+\sqrt{q}}$$

by multiplying both numerator and denominator by $\sqrt{p}-\sqrt{q}$. This process is called **rationalising the denominator**. Here are some examples for you to try.

### *Activity 1.5 Rationalising the denominator*

Rationalise the denominator of each of the following fractions.

(a) $\dfrac{1}{\sqrt{5}+\sqrt{2}}$ (b) $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$

Solutions are given on page 40.

The properties $\phi + \psi = 1$ and $\phi\psi = -1$ of $\phi$ and $\psi$ are special cases of two general properties of the solutions of a quadratic equation.

**Properties of solutions of quadratic equations**

Let $\alpha$ and $\beta$ be real solutions of the quadratic equation

$$ax^2 + bx + c = 0.$$

Then

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}.$$

For the golden ratio equation: $a = 1, b = -1$ and $c = -1$.

Therefore, we can write down the sum and product of real solutions of a quadratic equation, *without solving the equation*. For example, if $\alpha$ and $\beta$ are the solutions of $x^2 + 3x + 2 = 0$, then

$$\alpha + \beta = -\frac{3}{1} = -3 \quad \text{and} \quad \alpha\beta = \frac{2}{1} = 2.$$

The above two properties can be proved by using the formula for the solutions of the quadratic equation. The proof of the first property is given here:

$$\begin{aligned}\alpha+\beta &= \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)+\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)\\ &= \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}\\ &= \frac{-2b}{2a}=-\frac{b}{a},\end{aligned}$$

as required.

Further properties of the solutions $\alpha$ and $\beta$ can be derived from the equations $\alpha+\beta=-b/a$ and $\alpha\beta=c/a$. For example, the sum of the reciprocals of $\alpha$ and $\beta$ is:

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{-b/a}{c/a}=-\frac{b}{c}.$$

In particular, for the golden ratio equation:

$$\frac{1}{\phi}+\frac{1}{\psi}=-\frac{-1}{-1}=-1.$$

### *Activity 1.6 Finding sums and products of solutions*

For each of the following quadratic equations, find the sum of the solutions, the product of the solutions, and the sum of the reciprocals of the solutions, without solving the equation.

(a) $x^2-5x+6=0$ (b) $3x^2+x-1=0$

(c) $5x^2-2=0$.

Solutions are given on page 40.

Another way to find properties of a solution of an equation, without solving the equation, is to substitute the solution into the equation and then apply suitable algebraic rearrangements. For example, we know that $\phi=\frac{1}{2}(1+\sqrt{5})=1.618\ldots$ is a solution of the golden ratio equation $x^2-x-1=0$, so

$$\phi^2-\phi-1=0.$$

Writing this in the equivalent form

$$\phi^2=\phi+1$$

and then dividing both sides by $\phi$, we obtain the property

$$\phi=1+1/\phi. \tag{1.2}$$

Similar reasoning shows that $\psi=1+1/\psi$.

### *Activity 1.7 Another property of $\phi$*

Use the golden ratio equation to show that

$$\phi^3=2\phi+1.$$

A solution is given on page 40.

***Comment***

Similar reasoning shows that $\psi^3=2\psi+1$.

## 1.3 Other occurrences of the golden ratio

This subsection indicates some other contexts in which the golden ratio appears. These examples explain why much interest has been shown in this number over the years.

This subsection will not be assessed.

### *The golden ratio and the regular pentagon*

A **regular pentagon** is a pentagon with five equal sides and five equal angles. It is a remarkable fact, known to the Ancient Greeks, that the golden ratio appears in such a pentagon. Indeed, if the length of the sides of a regular pentagon is 1, then the length of each of its five diagonals is $\phi$, as is now shown.

It is possible that Pythagoras knew this as early as 500BC.

Figure 1.4 shows a regular pentagon $ABCDE$, two of whose diagonals $AC$ and $BE$ meet at $F$. The length of the sides is 1 and the length of the diagonals is denoted by $d$. Here is an outline proof that $d = \phi$.

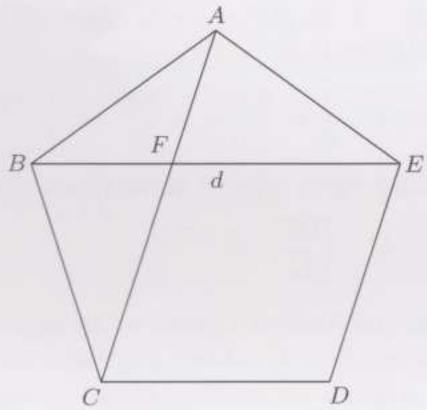


*Figure 1.4* Regular pentagon

The symmetric nature of the regular pentagon implies that $BE$ and $CD$ are parallel, as are $AC$ and $ED$. Thus $EDCF$ is a parallelogram, so the opposite sides $CD$ and $FE$ are of equal length. Hence $FE = CD = 1$, so $BF = d - 1$.

Similarly, $AF = d - 1$, so triangle $FAB$ is isosceles. Also, triangle $ABE$ is isosceles. Therefore triangles $FAB$ and $ABE$ are similar, since they have angle $ABF$ in common. Hence

$$\frac{BF}{BA} = \frac{BA}{BE}; \quad \text{that is,} \quad \frac{d-1}{1} = \frac{1}{d}.$$

This gives the golden ratio equation $d^2 - d - 1 = 0$. Since $d$ is positive, we deduce that $d = \phi$.

In fact, the number $\phi$ makes several appearances in the regular pentagon and in a number of other geometric figures. Suppose that all the diagonals

This star, called a *pentagram*, was the Pythagorean symbol of good health. It is a significant symbol for many communities.

of a regular pentagon $ABCDE$ are drawn to form a five-pointed star (Figure 1.5) and let $F$ and $G$ denote the points where $BE$ meets $AC$ and $AD$, respectively.

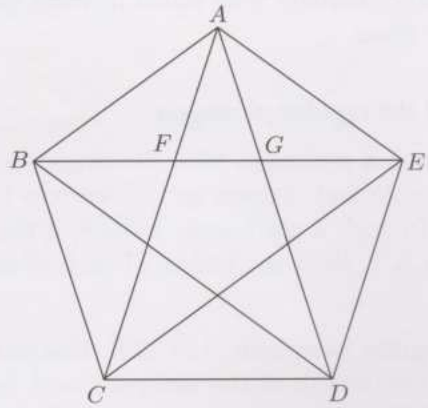


*Figure 1.5* Pentagram

Then $BE = \phi$ and $FE = 1$, so $BF = \phi - 1$. Thus $BF = 1/\phi$, by equation (1.2). Also, $BG = 1$, so

$$FG = BG - BF = 1 - \frac{1}{\phi} = \frac{\phi - 1}{\phi} = \frac{1/\phi}{\phi} = \frac{1}{\phi^2}.$$

It follows that the golden ratio appears three times along the diagonal $BE$:

$$\frac{BE}{BG} = \phi, \quad \frac{BG}{BF} = \phi, \quad \frac{BF}{FG} = \phi.$$

These observations led the Ancient Greeks to an ingenious demonstration that $\phi$ is an irrational number. If $\phi$ were a rational number, then we could write $\phi = p/q$, where $p$ and $q$ are positive integers. Since $\phi$ is greater than 1, we have $p > q$. By equation (1.2),

$$\frac{1}{\phi} = \phi - 1 = \frac{p}{q} - 1 = \frac{p - q}{q},$$

so

$$\phi = \frac{q}{p - q}.$$

But this last equation expresses $\phi$ as a fraction with numerator $q$, which is *smaller* than $p$. Using this reasoning again and again, we can express $\phi$ as a fraction with smaller and smaller numerator as often as we please. But, since these numerators are positive integers, this *cannot* happen indefinitely often. Hence $\phi$ must be irrational!

This reasoning is an example of 'proof by contradiction'.

Somewhat similar reasoning can be used to show that the numbers $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \ldots$, which are of the form $\sqrt{n}$, where $n$ is not a perfect square, are all irrational.

The perfect squares are: 1, 4, 9, 16, ....

### *The golden ratio in human artifacts*

There are many claims that the golden ratio has been used deliberately as an aesthetically pleasing proportion in the design of artifacts, ranging from Ancient Greek temples and sculptures, to the more recent examples below. These claims should be treated with caution; some uses of the golden ratio are certainly deliberate, but others have probably arisen by chance and are only approximate.

◇ Salvador Dali's painting *The Sacrament of the Last Supper* is painted in a golden rectangle, and the painting includes part of a huge dodecahedron, a three-dimensional solid whose faces are regular pentagons.

- From the mid-18th to the mid-19th century, many pieces of classical music had their main climax about 61% of the way through in order to give the music a kind of 'balance'. Later composers, such as Bartok and Debussy, are known to have used the golden ratio deliberately.
- The architect Le Corbusier wrote extensively about a system of design based on the golden ratio and used it in some of his buildings.
- The design of one of the rose windows in Chartres cathedral appears to be based on the golden ratio.
- In Leo Tolstoy's long novel *Anna Karenina* only one chapter has a title, which is *Death*, and this occurs about 61% of the way through the book.

## Summary of Section 1

This section has introduced:

- several geometric situations that give rise to the golden ratio $\phi$, the positive solution of the golden ratio equation

  $$x^2 - x - 1 = 0;$$

- some common features of problem solving in mathematics;
- the following properties of real solutions $\alpha$ and $\beta$ of the quadratic equation $ax^2 + bx + c = 0$:

  $$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a};$$

- some properties of the solutions of the golden ratio equation

  $$\phi = \tfrac{1}{2}(1 + \sqrt{5}) = 1.618\ldots \quad \text{and} \quad \psi = \tfrac{1}{2}(1 - \sqrt{5}) = -0.618\ldots,$$

  including

  $$\phi + \psi = 1, \quad \phi\psi = -1, \quad \phi = 1 + \frac{1}{\phi}, \quad \psi = 1 + \frac{1}{\psi};$$

- some examples of possible uses of the golden ratio.

## Exercises for Section 1

### Exercise 1.1

Suppose that a rectangle has two squares removed, one from each end, leaving a rectangle the same shape as the original rectangle. What is the ratio of the lengths of the sides of the original rectangle?

### Exercise 1.2

For each of the following quadratic equations, find the sum of the solutions, the product of the solutions, and the sum of the squares of the solutions, without solving the equation. (Use the identity $(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$.)

(a) $x^2 - 4x + 3 = 0$

(b) $4x^2 - 2x - 1 = 0$

(c) $3x^2 - 2 = 0$.

### Exercise 1.3

(a) Use the fact that $\phi$ and $\psi$ satisfy the golden ratio equation, together with the fact that $\phi + \psi = 1$, to evaluate each of the expressions

$$\phi^n + \psi^n \quad (n = 0, 1, \ldots, 4).$$

For example, for $n = 0$,

$$\phi^0 + \psi^0 = 1 + 1 = 2.$$

(b) Do you detect a pattern in your answers?

# 2 Fibonacci numbers and their properties

In this section, we consider a remarkable sequence of positive integers which, like the golden ratio, has a long history and occurs unexpectedly in many places. We explore several properties of this sequence and find that it appears to be related to the golden ratio.

## 2.1 What are Fibonacci numbers?

Leonardo of Pisa (about 1170–1250), was one of the greatest mathematicians of the Middle Ages. He was educated in North Africa, where his father worked in commerce.

In 1202 Leonardo of Pisa published *Liber abaci* (The book of calculations). This described methods of arithmetic and algebra from the Arabic world, and advocated using the Hindu-Arabic numerals $0, 1, 2, \ldots, 9$, to represent numbers in positional notation. The book contained many problems, including this one about breeding rabbits.

**The Rabbit Problem**

Suppose that one pair of adult rabbits is placed in isolation and that

- ◇ the initial pair of adult rabbits produces a new pair in the first month, and in each subsequent month;
- ◇ each new pair becomes adult after one month and produces its first new pair in its second month and a new pair in each subsequent month;
- ◇ no rabbits die.

How many pairs of rabbits are there after a year?

This is not a realistic problem about rabbit breeding, but the description does give it a memorable context. Indeed, Leonardo could scarcely have guessed that he would be remembered for his rabbit problem, though under the name Fibonacci by which he became known.

Fibonacci is thought to be a contraction of *filius Bonacci*; that is, 'son of the Bonacci family'.

Consider how the number of pairs of rabbits develops month by month. One new pair is born in the first month, so there are two pairs. In month 2, the original adult pair produces another pair, so there are then 3 pairs. In month 3, the original adult pair produces yet another pair and the pair born in month 1 also produce a pair, making 5 pairs in all. This development is shown in Figure 2.1, where adult pairs are shown as solid circles and new pairs as empty circles.

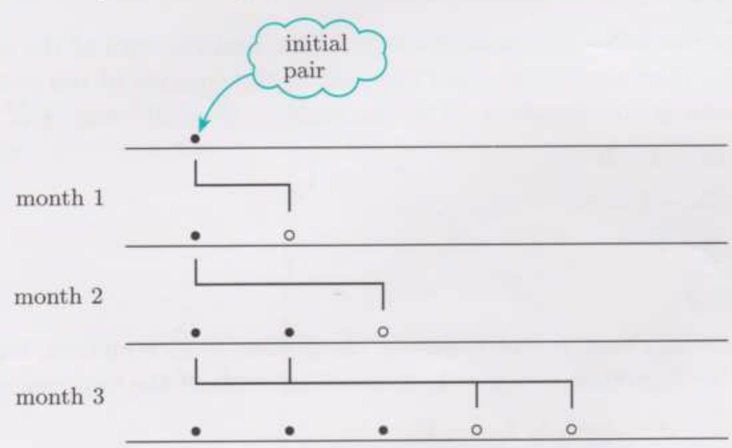


*Figure 2.1* Total number of pairs at ends of months

In general, we see that at the end of each month:

◇ the number of adult pairs is equal to the total number of pairs at the end of the previous month,

◇ the number of new pairs is equal to the number of adult pairs at the end of the previous month.

Using these two observations, we obtain the results in Table 2.1.

*Table 2.1* Rabbit pairs at the end of each month

| Month | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Adult pairs | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | 233 |
| New pairs | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 |
| Total pairs | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | 233 | 377 |

The entry in the bottom right of this table shows that the answer to Leonardo's question is 377, which is not especially interesting in itself. The number patterns in the table are striking, however.

The final three rows of the table all contain essentially the same sequence of positive integers $1, 1, 2, 3, 5, 8, 13, \ldots$, each term of which (after the second) is the sum of the two previous terms:

$$2 = 1 + 1,\ 3 = 2 + 1,\ 5 = 3 + 2,\ 8 = 5 + 3,\ 13 = 8 + 5, \ldots.$$

The terms of this sequence were called **Fibonacci numbers** by the French mathematician Edouard Lucas in 1877. It is common to include an extra term at the beginning of the sequence, as follows:

Lucas (1842–1891) used Fibonacci numbers to give a remarkable proof that $2^{127} - 1$ is a prime number; that is, it has no factors other than 1 and itself.

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots.$$

With this extra term, it is still true that each term (after the second) is the sum of the previous two terms. We use the notation $F_n$ for Fibonacci numbers, starting with $F_0 = 0$ and $F_1 = 1$. Thus the sequence $F_n$, the **Fibonacci sequence**, is specified as a recurrence system:

For example, for $n = 0$,

$$F_2 = F_1 + F_0.$$

$$F_0 = 0, F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n = 0, 1, 2, \ldots).$$

The Fibonacci numbers $F_2, F_3, \ldots, F_{14}$ can be read off from Table 2.1.

*Table 2.2* Fibonacci numbers

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_n$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | 233 | 377 | | |

A recurrence relation such as $F_{n+2} = F_{n+1} + F_n$, where each term depends on the previous two terms, is called a **second-order recurrence relation** and the corresponding recurrence system is called a **second-order recurrence system**. To specify a second-order recurrence system, we need *two* initial terms. As with first-order recurrence systems, we can set out the recurrence system in other ways, such as

$$F_0 = 0, F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \quad (n = 1, 2, 3, \ldots).$$

You are now asked to calculate two more Fibonacci numbers.

### *Activity 2.1 More Fibonacci numbers*

Use the Fibonacci recurrence relation to complete Table 2.2.

A solution is given on page 40.

Fibonacci numbers have many remarkable algebraic properties, some of which are studied in the next two activities.

### *Activity 2.2 Fibonacci sums*

For example, for $n = 2$,

$$F_1 + F_2 = 1 + 1 = 2.$$

(a) Calculate each of the sums

$$F_1 + F_2 + \cdots + F_n \quad (n = 1, 2, \ldots, 8).$$

(b) Do you detect a pattern in your answers to part (a)? If so, what general property of Fibonacci numbers is suggested by this pattern?

Solutions are given on page 40.

### *Activity 2.3 Fibonacci ratios*

For example, for $n = 2$,

$$\frac{F_3}{F_2} = \frac{2}{1} = 2.$$

(a) Calculate each of the ratios

$$\frac{F_{n+1}}{F_n} \quad (n = 1, 2, \ldots, 8),$$

to three decimal places.

(b) Do you detect a pattern in your answers to part (a)? If so, what general property of Fibonacci numbers is suggested by this pattern?

Solutions are given on page 41.

These activities illustrate a fundamental aspect of mathematical exploration that occurs often. When calculations suggest the presence of a pattern, it is natural to suspect that there is some underlying reason for the pattern. This may lead us to formulate a general statement which, if true, explains the pattern. Such a general statement, which we feel may be true, but of which we have no proof, is called a **conjecture**.

For example, in Activity 2.2 you found that the first eight Fibonacci sums have the values in the following table.

*Table 2.3* Fibonacci sums

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $F_n$ | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 |
| $F_1 + F_2 + \cdots + F_n$ | 1 | 2 | 4 | 7 | 12 | 20 | 33 | 54 |

Each of these sums is one less than the Fibonacci number two places to the right and, on this basis, it is natural to make the following conjecture.

**Conjecture 1**
For $n = 1, 2, 3 \dots,$

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1.$$

For example, for $n = 3$,

$$F_1 + F_2 + F_3 = 1 + 1 + 2 = 4$$

and

$$F_5 - 1 = 5 - 1 = 4.$$

This conjecture is appealingly simple to state, but perhaps it is 'too good to be true'. The sceptical approach to such a conjecture is to consider further Fibonacci sums, and try to find an integer $n$ for which the equation is false. Such an example, which shows that a conjecture is false, is called a **counter-example**. If no counter-example can be found, then the sceptic may be persuaded that the conjecture is perhaps true and try to find a proof of it. Note, however, that Conjecture 1 is a statement about all positive integers $n$, so you cannot *prove* it by checking just a finite number of cases, no matter how large that number might be.

In a similar way, the calculation in Activity 2.3 leads to the following conjecture.

**Conjecture 2**
The terms of the sequence $F_{n+1}/F_n$ $(n = 1, 2, 3, \dots)$ lie alternately above and below the golden ratio $\phi$, and

$$\frac{F_{n+1}}{F_n} \to \phi \text{ as } n \to \infty.$$

The pattern leading to this conjecture is particularly clear if we plot a graph of the sequence of ratios $F_{n+1}/F_n$ $(n = 1, 2, 3, \dots)$.

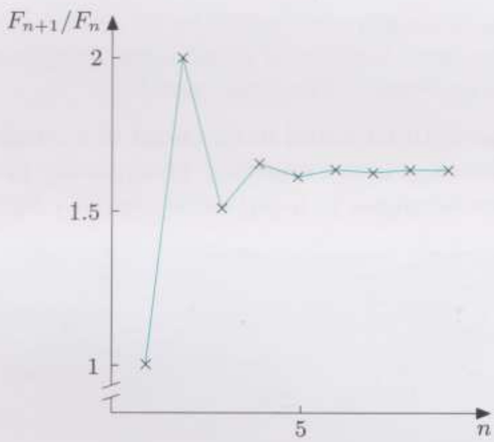


*Figure 2.2* Ratios of successive pairs of Fibonacci numbers

Further study of Table 2.2 reveals more patterns. For example, these Fibonacci numbers are even or odd according to the following pattern:

even, odd, odd, even, odd, odd, ....

This pattern arises because 'odd + odd = even' and 'odd + even = odd'. On checking factors of these Fibonacci numbers in more detail, we find a further pattern:

$F_3$ is a factor of $F_3, F_6, F_9, F_{12}, F_{15}$;
$F_4$ is a factor of $F_4, F_8, F_{12}, F_{16}$;
$F_5$ is a factor of $F_5, F_{10}, F_{15}$.

On this basis, it is natural to make the following conjecture.

**Conjecture 3**
If $n$ is a factor of $m$, then $F_n$ is a factor of $F_m$.

By convention, the number 1 is not included amongst the prime numbers.

Now, consider the values of $n$ in Table 2.2 which are odd prime numbers: $n = 3, 5, 7, 11, 13$. It is not hard to check that all the corresponding Fibonacci numbers

$$F_3 = 2, F_5 = 5, F_7 = 13, F_{11} = 89, F_{13} = 233,$$

are themselves prime. On this basis, it is natural to make the following conjecture.

**Conjecture 4**
If $n$ is an odd prime number, then $F_n$ is a prime number.

Later in the chapter, you will see which of the above conjectures are true and which are false.

## 2.2 Fibonacci numbers in nature

This subsection will not be assessed.

In this subsection, some occurrences of Fibonacci numbers in nature are briefly described, and a possible explanation for these is presented.

It has been known for centuries that Fibonacci numbers occur in the structure of plants. For example:

- ◇ the number of petals on many flowers is 3, 5, 8, 13, 21, or 34;
- ◇ in many plants, the successive leaves along a stem are rotated around the stem by the same fraction of a full turn, and these fractions are often ratios of consecutive Fibonacci numbers.

A dramatic example is to be found in the head of a sunflower. This head consists of many florets, which eventually produce the sunflower's seeds, and these florets are arranged in a distinctive spiral pattern.

Similar patterns can be seen in dahlia flowers, pineapples, fir cones and cauliflowers!

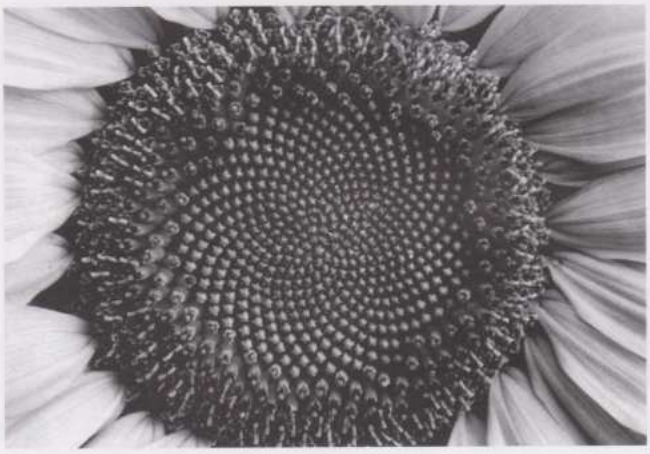

*Figure 2.3* Sunflower head (© Dwight Kuhn)

In Figure 2.3, several sets of spirals can be seen – sets of 21 and 55 spirals in one direction and 34 in the other direction. It is common for the numbers of such spirals to be consecutive Fibonacci numbers.

Why should Fibonacci numbers occur in this way? No definitive answer has been given but one possible explanation, described below, involves the laws of physics, natural selection and the golden ratio!

More detail about this explanation can be found in *Nature's Numbers* by Ian Stewart. (Stewart, I (1998) *Nature's Numbers*, Phoenix, a division of Orion Books Ltd.)

The florets of a sunflower develop from tiny lumps called *primordia*. Primordia are created one after another on the edge of a central disc, called the *apex*, and then move outwards. The angle between the points on the apex at which successive new primordia appear is found in most sunflowers to be approximately 137.5°. After some time, the primordia are located as in Figure 2.4, in which the labels $1, 2, 3, \ldots$ refer to the order in which the primordia were created.

Primordia may appear in clockwise or anticlockwise order.

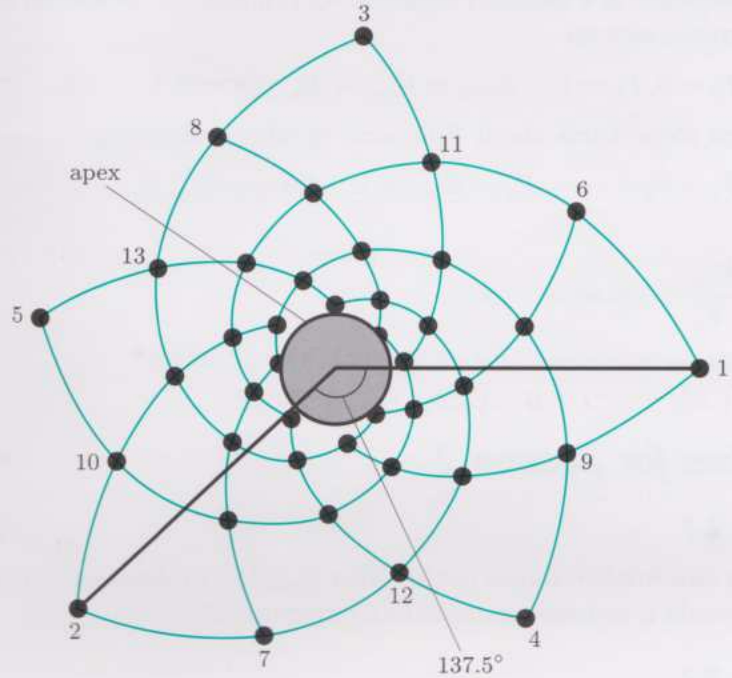


*Figure 2.4* Location of primordia

Figure 2.4 also shows the spirals formed by nearby primordia. The fact that the numbers of these spirals are Fibonacci numbers results from two mathematical facts.

◇ A connection between 137.5° and the golden ratio $\phi$, namely, that:

$$360 - 137.5\ (= 222.5) \text{ is very close to } \frac{360}{\phi}\ (= 222.492\ldots).$$

◇ The ratios of consecutive Fibonacci numbers

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \ldots,$$

are good approximations to $1/\phi = 0.618\ldots$.

You saw in Activity 2.3 that the sequence of reciprocals of these ratios are good approximations to $\phi$.

Which of these Fibonacci numbers are visible as numbers of spirals seems to depend on the rates at which the primordia are created and move away from the apex.

But why has nature chosen the angle between the successive primordia to be 137.5°, rather than something simple like 120°? An angle which *is* a simple fraction of 360° would leave the primordia arranged on a small number of radial lines (three in the case of 120°), and this does not result in an efficient use of the available space for large numbers of seeds to develop. On the other hand, in a certain technical sense the angle 137.5° is

about as far as possible from being a simple fraction of 360°, and so leads to the most efficient packing of the head by seeds.

The exact mechanism by which the angle 137.5° arises is not yet fully understood.

In summary, the presence of Fibonacci numbers of spirals in the sunflower head *seems* to be a mathematical consequence of nature selecting a good approximation to the golden ratio in order to pack sunflower seeds as efficiently as possible.

## Summary of Section 2

This section has introduced:

◇ the sequence of Fibonacci numbers, $0, 1, 1, 2, 3, 5, \ldots$, defined by the recurrence system

$$F_0 = 0, F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n = 0, 1, 2, \ldots);$$

◇ several conjectures about Fibonacci numbers, including

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1, \quad \text{for } n = 1, 2, 3 \ldots,$$

and

$$\frac{F_{n+1}}{F_n} \to \phi \text{ as } n \to \infty;$$

◇ several occurrences of Fibonacci numbers in nature.

## Exercises for Section 2

### Exercise 2.1

Calculate two further values of the ratio $F_{n+1}/F_n$ in Activity 2.3. Do the values provide a counter-example to Conjecture 2?

### Exercise 2.2

(a) Calculate the values of each of the expressions

$$F_{n-1}F_{n+1} - F_n^2 \quad (n = 1, 2, \ldots, 6).$$

(b) Try to detect a pattern in your answers to part (a) and state a conjecture suggested by this pattern.

### Exercise 2.3

(a) Calculate the values of each of the expressions

$$F_1^2 + F_2^2 + \cdots + F_n^2 \quad (n = 1, 2, \ldots, 6).$$

(b) Try to detect a pattern in your answers to part (a) and state a conjecture suggested by this pattern.

# 3 Linear second-order recurrence sequences

To study Subsection 3.1 you will need a cassette player and Audio Tape 1.

In this section, we consider a certain type of second-order recurrence system, and find a remarkable formula for Fibonacci numbers.

## 3.1 What is a linear second-order recurrence sequence?

The Fibonacci recurrence relation $F_{n+2} = F_{n+1} + F_n$ is a special case of a general type of recurrence relation of the form

$$u_{n+2} = pu_{n+1} + qu_n,$$

where the coefficients $p$ and $q$ are constants, and $q \neq 0$.

For the Fibonacci recurrence relation, $p = 1$ and $q = 1$.

The full name of such recurrence relations is **linear second-order constant-coefficient homogeneous recurrence relations**, but we shall usually omit the words 'constant-coefficient' and 'homogeneous'. When two initial terms, such as $u_0$ and $u_1$, are also specified, we refer to a **linear second-order recurrence system**, set out in this manner:

$$u_0 = a, u_1 = b, \quad u_{n+2} = pu_{n+1} + qu_n \quad (n = 0, 1, 2, \ldots). \tag{3.1}$$

An *inhomogeneous* recurrence relation includes expressions other than terms of the sequence; an example is

$$u_{n+2} = u_{n+1} + u_n + 2n.$$

Here $a$ and $b$ are the values of the two initial terms. A sequence which arises in this way is called a **linear second-order recurrence sequence**.

It is not clear from this definition whether, for every linear second-order recurrence sequence, there is a formula for $u_n$ which enables us to calculate any term of the sequence directly once we are given the value of $n$. Such a formula is called a **closed form** (or **closed-form formula**). A clue to finding a closed form is provided by Conjecture 2 in Section 2, which includes the property that

$$\frac{F_{n+1}}{F_n} \to \phi \text{ as } n \to \infty,$$

In MST121, Chapter A1, we obtained closed forms for various sequences defined by first-order recurrence systems.

where $\phi$ is the golden ratio. If true, this property suggests that, for large values of $n$, $F_{n+1}$ is close to $\phi F_n$, so the sequence $F_n$ is 'nearly' geometric. This in turn suggests that we start to seek a closed form for equation (3.1) by trying a geometric sequence. We do this in the following audio tape. To prepare for this tape, you should first try the following activity, the solutions to which appear in the tape frames.

Geometric sequences were introduced in MST121, Chapter A1, Section 3.

### Activity 3.1 Two quadratic equations

Solve each of the following quadratic equations.

(a) $r^2 - 12r + 20 = 0$ (b) $r^2 - 4r + 4 = 0$

Solutions are contained in Frames 2 and 4.

***Now listen to Audio Tape 1, Band 1, 'Linear second-order recurrence sequences', while you study the tape frames.***

## Frame 1

**Example 1**

$u_{n+2} = 12u_{n+1} - 20u_n \quad (n = 0, 1, 2, \ldots)$

Equation (3.1) with $p = 12, q = -20$

Complete the table for each of the following pairs of initial terms.

| | $u_0$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ | ... | $u_n$ |
|---|---|---|---|---|---|---|---|
| (a) | 1 | 2 | 4 | 8 | 16 | | $2^n$ |
| (b) | 1 | 10 | 100 | 1,000 | 10,000 | | $10^n$ |
| (c) | 2 | 12 | 104 | 1,008 | 10,016 | | $2^n + 10^n$ |
| (d) | 3 | 22 | 204 | 2,008 | 20,016 | | $2^n + 2(10^n)$ |

Closed form conjectures

## Frame 2

**A closed form for Example 1(d)**

Find a closed form for:

$$u_0 = 3, u_1 = 22, \quad u_{n+2} = 12u_{n+1} - 20u_n \; (*) \quad (n = 0, 1, 2, \ldots).$$

Try a geometric sequence, $u_n = r^n$ $(r \neq 0)$:

$$r^{n+2} = 12r^{n+1} - 20r^n \quad \text{(substitute in } (*)\text{)},$$

So $u_{n+2} = r^{n+2}$, $u_{n+1} = r^{n+1}$.

so

$$r^2 = 12r - 20 \quad \text{(divide through by } r^n\text{)};$$

that is,

$$r^2 - 12r + 20 = 0,$$

Auxiliary equation — Activity 3.1

which has solutions $r = 2$ and $r = 10$. Thus

$$u_n = 2^n \quad \text{and} \quad u_n = 10^n \text{ satisfy } (*).$$

General solution of $(*)$:

$$u_n = A2^n + B10^n.$$

A, B unknown constants

Use initial terms to find A and B:

$$\left.\begin{array}{lll} n=0 & u_0 = 3 & A + B = 3 \\ n=1 & u_1 = 22 & 2A + 10B = 22 \end{array}\right\} \text{give } A = 1, B = 2.$$

$2^0 = 1$, $10^0 = 1$.

Closed form:

$$u_n = 2^n + 2 \times 10^n \quad (n = 0, 1, 2, \ldots).$$

## Frame 3

**Example 2**

$u_{n+2} = 4u_{n+1} - 4u_n \quad (n = 0, 1, 2, \ldots)$

Equation (3.1) with $p = 4, q = -4$

Complete the table for each of the following pairs of initial terms.

| | $u_0$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ | ... | $u_n$ |
|---|---|---|---|---|---|---|---|
| (a) | 1 | 2 | 4 | 8 | 16 | | $2^n$ |
| (b) | 0 | 2 | 8 | 24 | 64 | | $n \times 2^n$ |
| (c) | 3 | 8 | 20 | 48 | 112 | | $3 \times 2^n + n \times 2^n$ |

Closed form conjectures

## Frame 4

**A closed form for Example 2(c)**

Find a closed form for:

$$u_0 = 3,\ u_1 = 8, \quad u_{n+2} = 4u_{n+1} - 4u_n \ (*) \quad (n = 0, 1, 2, \ldots).$$

Try $u_n = r^n$ $(r \neq 0)$:

$$r^{n+2} = 4r^{n+1} - 4r^n \quad \text{(substitute in (*))},$$

so

$$r^2 = 4r - 4 \quad \text{(divide through by } r^n\text{)};$$

Auxiliary equation — Activity 3.1

that is,

$$r^2 - 4r + 4 = 0,$$

which has solution $r = 2$ (repeated). Thus
$u_n = 2^n$ satisfies (*).

Example 2(a)

<u>Assume</u> that $u_n = n2^n$ also satisfies (*).

Example 2(b)

General solution of (*):

$$u_n = A2^n + Bn2^n.$$

Use initial terms to find A and B:

$$\left.\begin{array}{lll} n = 0 & u_0 = 3 & A = 3 \\ n = 1 & u_1 = 8 & 2A + 2B = 8 \end{array}\right\} \text{ give } A = 3, B = 1.$$

Closed form:

$$u_n = 3 \times 2^n + n2^n \quad (n = 0, 1, 2, \ldots).$$

## Frame 5

### Strategy

To find a closed form for a linear second-order recurrence system:

$$u_0 = a,\ u_1 = b,\quad \boxed{u_{n+2} = pu_{n+1} + qu_n\ (*)}\quad (n = 0, 1, 2, \ldots).$$

Step 1 Write down the auxiliary equation:

$r^2 - pr - q = 0.$

Try $u_n = r^n$.

Step 2 Solve the auxiliary equation:

$r = \alpha$ and $r = \beta$.

Step 3 Write down the general solution, with unknown constants A and B.

| | |
|---|---|
| $\alpha, \beta$ real and distinct | $u_n = A\alpha^n + B\beta^n$ |
| $\alpha = \beta$, repeated root | $u_n = (A + Bn)\alpha^n$ |
| no real solutions | more complicated |

Powers, sines and cosines

Step 4 Use initial terms $u_0 = a$, $u_1 = b$ to find A and B.

## Frame 6

### The repeated root case

Why does $u_n = n\alpha^n$ satisfy (*) in Frame 5 if $\alpha$ is a repeated root?

Auxiliary equation: $(r - \alpha)^2 = r^2 - 2\alpha r + \alpha^2 = 0.$

$p = 2\alpha,\ q = -\alpha^2.$

Recurrence relation: $u_{n+2} = 2\alpha u_{n+1} - \alpha^2 u_n.$ (†)

Check:

RHS of (†) $= 2\alpha(n+1)\alpha^{n+1} - \alpha^2 n\alpha^n$

$= 2n\alpha^{n+2} + 2\alpha^{n+2} - n\alpha^{n+2}$

$= (n+2)\alpha^{n+2} = u_{n+2} =$ LHS of (†).

Substitute $u_n = n\alpha^n$.

✓

### *Activity 3.2 Finding closed forms*

Use the strategy in Frame 5 to find a closed form for each of the following sequences.

(a) $u_0 = 2, u_1 = 7, \quad u_{n+2} = 7u_{n+1} - 12u_n \quad (n = 0, 1, 2, \ldots)$

(b) $u_0 = 2, u_1 = 7, \quad u_{n+2} = 6u_{n+1} - 9u_n \quad (n = 0, 1, 2, \ldots)$

(c) $u_0 = 2, u_1 = 7, \quad u_{n+2} = 2u_{n+1} + 8u_n \quad (n = 0, 1, 2, \ldots)$

Solutions are given on page 41.

## 3.2 A formula for Fibonacci numbers

Since the Fibonacci sequence is defined by the linear second-order recurrence system

$$F_0 = 0, F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n = 0, 1, 2, \ldots),$$

we can find a closed form for Fibonacci numbers by using the strategy in the audio tape. The first step is to write down the auxiliary equation of the recurrence relation $u_{n+2} = u_{n+1} + u_n$:

$$r^2 - r - 1 = 0.$$

This is the golden ratio equation, which has the two solutions

$$r = \phi = \tfrac{1}{2}(1 + \sqrt{5}) = 1.618\ldots \quad \text{and} \quad r = \psi = \tfrac{1}{2}(1 - \sqrt{5}) = -0.618\ldots.$$

See Activity 1.2.

Thus the general solution of the recurrence relation is

$$u_n = A\phi^n + B\psi^n,$$

where $A$ and $B$ are unknown constants. In particular, the Fibonacci numbers are of this form with constants $A$ and $B$ determined by the initial terms $u_0 = F_0 = 0$ and $u_1 = F_1 = 1$:

$$F_0 = 0 \quad \text{gives} \quad A + B = 0,$$

$$F_1 = 1 \quad \text{gives} \quad A\phi + B\psi = 1.$$

On substituting $B = -A$ from the first equation into the second and then using the fact that $\phi - \psi = \sqrt{5}$, we obtain

See Activity 1.4 (b).

$$A\phi - A\psi = 1,$$

so

$$A = \frac{1}{\phi - \psi} = \frac{1}{\sqrt{5}}.$$

Thus $B = -A = -1/\sqrt{5}$, so the required closed form is

$$F_n = \frac{1}{\sqrt{5}}\phi^n - \frac{1}{\sqrt{5}}\psi^n = \frac{1}{\sqrt{5}}(\phi^n - \psi^n) \quad (n = 0, 1, 2, \ldots).$$

This remarkable formula for Fibonacci numbers was first obtained in the early 18th century, and was rediscovered in 1843 by J.P.M.Binet, after whom it is named. The formula is remarkable because it involves three *irrational* numbers $\sqrt{5}, \phi$ and $\psi$, yet it gives an integer for every value of $n$.

Jacques Philippe Marie Binet (1786–1856) was a French mathematician and astronomer.

**Binet's formula**

A closed form for the Fibonacci numbers is

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - \psi^n) \quad (n = 0, 1, 2, \ldots),$$

where $\phi = \frac{1}{2}(1+\sqrt{5})$ and $\psi = \frac{1}{2}(1-\sqrt{5})$.

The two sequences $\phi^n$ and $\psi^n$ which appear in Binet's formula have different long-term behaviour. Since $\phi = 1.618\ldots$ is greater than 1, the geometric sequence $\phi^n$ tends to infinity as $n$ tends to infinity. On the other hand, $\psi = -0.618\ldots$ lies between $-1$ and 0, so the sequence $\psi^n$ alternates in sign and tends to 0 as $n$ tends to infinity. For example, to three significant figures,

See MST121, Chapter A1, Section 5, for a discussion of these types of long-term behaviour.

$$\psi = -0.618,\ \psi^2 = 0.382,\ \psi^3 = -0.236,\ \ldots,\ \psi^{15} = -7.33 \times 10^{-4}.$$

Binet's formula is not suitable for hand-calculation of Fibonacci numbers, other than for small $n$ for which the values of $F_n$ are already known. It is suitable for finding Fibonacci numbers with a calculator, however, particularly if the following version of the formula is used.

**Binet's approximation**

For $n = 0, 1, 2, \ldots$, the Fibonacci number $F_n$ lies within $\frac{1}{2}$ of $\phi^n/\sqrt{5}$, so

$$F_n \text{ is the nearest integer to } \frac{\phi^n}{\sqrt{5}}.$$

In symbols,

$$-\tfrac{1}{2} < F_n - \phi^n/\sqrt{5} < \tfrac{1}{2}.$$

This holds because, for $n = 0, 1, 2, \ldots,$

$$\psi^n \text{ lies between } -1 \text{ and } 1,$$

so, since $\sqrt{5} = 2.236\ldots > 2$,

$$\frac{\psi^n}{\sqrt{5}} \text{ lies between } -\frac{1}{2} \text{ and } \frac{1}{2},$$

and hence, by Binet's formula,

$$F_n \text{ lies within } \frac{1}{2} \text{ of } \frac{\phi^n}{\sqrt{5}}.$$

Since $F_n$ is an *integer*, it must be the nearest integer to $\phi^n/\sqrt{5}$.

For example, with $n = 15$, a calculator gives

$$\frac{\phi^{15}}{\sqrt{5}} = \frac{(\frac{1}{2}(1+\sqrt{5}))^{15}}{\sqrt{5}} = 609.999\,67 \text{ (to 5 d.p.)},$$

so

$$F_{15} = 610.$$

### *Activity 3.3 Using Binet's approximation*

Use Binet's approximation to calculate $F_{20}$.

A solution is given on page 41.

## Summary of Section 3

This section has introduced:

◇ linear second-order recurrence sequences, defined by recurrence systems of the form:

$$u_0 = a, u_1 = b, \quad u_{n+2} = pu_{n+1} + qu_n \quad (n = 0, 1, 2, \ldots),$$

and a strategy to find a closed form for such sequences when the auxiliary equation

$$r^2 - pr - q = 0$$

has real solutions;

◇ Binet's formula for Fibonacci numbers:

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - \psi^n) \quad (n = 0, 1, 2, \ldots),$$

where $\phi = \frac{1}{2}(1 + \sqrt{5})$ and $\psi = \frac{1}{2}(1 - \sqrt{5})$;

◇ Binet's approximation:
for $n = 0, 1, 2, \ldots$, the Fibonacci number $F_n$ lies within $\frac{1}{2}$ of $\phi^n/\sqrt{5}$, so

$$F_n \text{ is the nearest integer to } \frac{\phi^n}{\sqrt{5}}.$$

## Exercises for Section 3

### Exercise 3.1

Find a closed form for each of the following sequences.

(a) $u_0 = 0, u_1 = \frac{5}{2}, \quad u_{n+2} = \frac{3}{2}u_{n+1} + u_n \quad (n = 0, 1, 2, \ldots)$

(b) $u_0 = 2, u_1 = 7, \quad u_{n+2} = 0.9u_{n+1} - 0.2u_n \quad (n = 0, 1, 2, \ldots)$

(c) $u_0 = -1, u_1 = 2, \quad u_{n+2} = -u_{n+1} - \frac{1}{4}u_n \quad (n = 0, 1, 2, \ldots)$

### Exercise 3.2

This exercise concerns the **Lucas numbers**, which satisfy the Fibonacci recurrence relation but have different initial terms:

$$L_0 = 2, L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n \quad (n = 0, 1, 2, \ldots).$$

(a) Calculate $L_2, L_3, \ldots, L_6$.

(b) Find a formula for the Lucas numbers.

(c) Use your formula to calculate $L_{15}$, making use of the fact that, for $n = 2, 3, 4, \ldots$,

$$-\tfrac{1}{2} < \psi^n < \tfrac{1}{2}.$$

# 4 Exploring linear second-order recurrence sequences with the computer

In this section you will need computer access, the files for this chapter and Computer Book A.

The computer can be used to obtain tables of many terms of linear second-order recurrence sequences (in particular the Fibonacci sequence) and of expressions related to such sequences. We can then study these tables to detect patterns that occur in such sequences.

*Refer to Computer Book A for the work in this section.*

## Summary of Section 4

This section has used the computer to help discover patterns occurring in linear second-order recurrence sequences and so form conjectures about the sequences on the basis of these patterns.

# 5 Proving identities

In this section, several conjectures that were made earlier about Fibonacci numbers are proved. Very many properties of Fibonacci numbers have been discovered over the centuries and the ones given here represent only a small selection.

There is an international journal devoted to properties of Fibonacci numbers and other related sequences.

Proofs in mathematics usually consist of a mixture of logical steps and intuitive insights (good ideas), and skill at producing proofs comes only with practice. The proofs in this section use various techniques, including

- ◇ the rules for rearranging algebraic expressions; (MST121, Chapter A0.)
- ◇ properties of $\phi$ and $\psi$; (Section 1)
- ◇ Binet's formula for Fibonacci numbers; (Subsection 3.2)
- ◇ the formula for the sum of a finite geometric series. (MST121, Chapter A1, Section 4)

These proofs are included to give you a first impression of what proofs are like. You will not be expected to produce such proofs yourself at this stage, except in certain straightforward cases.

For convenience, the properties of $\phi$ and $\psi$ which may be needed are repeated here.

**Properties of $\phi$ and $\psi$**

The numbers

$$\phi = \tfrac{1}{2}(1+\sqrt{5}) = 1.618\ldots \text{ and } \psi = \tfrac{1}{2}(1-\sqrt{5}) = -0.618\ldots,$$

satisfy:

$$\phi + \psi = 1 \text{ and } \phi\psi = -1; \qquad (5.1)$$

$$\phi - \psi = \sqrt{5}; \qquad (5.2)$$

$$\phi = 1 + 1/\phi \text{ and } \psi = 1 + 1/\psi. \qquad (5.3)$$

## 5.1 The Fibonacci sum identity

In Section 2, we considered the Fibonacci sums given in the following table.

*Table 5.1* Fibonacci sums

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $F_n$ | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 |
| $F_1 + F_2 + \cdots + F_n$ | 1 | 2 | 4 | 7 | 12 | 20 | 33 | 54 |

On the basis of these values, it was conjectured that each Fibonacci sum is one less than the Fibonacci number two places to the right. In fact, this conjecture is true, as you will see below.

Section 2, Conjecture 1

**Fibonacci sum identity**

For $n = 1, 2, 3, \ldots,$

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1.$$

Most true identities can be proved in several different ways. To illustrate this, two proofs of the above identity are given.

### Method 1: using the recurrence relation

The following approach is based on a 'good idea', namely, to rearrange the Fibonacci recurrence relation $F_{n+2} = F_{n+1} + F_n$ in the form

$$F_n = F_{n+2} - F_{n+1} \quad (n = 0, 1, 2, \ldots). \tag{5.4}$$

In particular, for $n = 0, 1, 2, \ldots$, the following $n + 1$ equations are true:

$$\begin{aligned}
F_0 &= F_2 - F_1,\\
F_1 &= F_3 - F_2,\\
F_2 &= F_4 - F_3,\\
&\ldots\\
F_n &= F_{n+2} - F_{n+1}.
\end{aligned} \tag{5.5}$$

Now you can see why this rearrangement of the recurrence relation is a good idea. If we *add* these equations, then:

- ◇ on the left-hand side, we obtain the sum $F_0 + F_1 + F_2 + \cdots + F_n$;
- ◇ on the right-hand side, all the terms cancel except $F_{n+2}$ and $-F_1$.

For example, $F_2$ in the first equation cancels $-F_2$ in the second.

Therefore, for $n = 1, 2, 3, \ldots$,

$$F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - F_1.$$

Since $F_0 = 0$ and $F_1 = 1$, this gives, for $n = 1, 2, 3, \ldots$,

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1,$$

which is the required result.

The type of repeated cancelling used to simplify the right-hand side above is called **telescoping cancellation**.

### Method 2: using Binet's formula

An alternative approach is to use Binet's formula to substitute for $F_1, F_2, \ldots, F_n$:

$$\begin{aligned}
F_1 + F_2 + \cdots + F_n &= \frac{1}{\sqrt{5}}(\phi - \psi) + \frac{1}{\sqrt{5}}(\phi^2 - \psi^2) + \cdots + \frac{1}{\sqrt{5}}(\phi^n - \psi^n)\\
&= \frac{1}{\sqrt{5}}\left((\phi + \phi^2 + \cdots + \phi^n) - (\psi + \psi^2 + \cdots + \psi^n)\right).
\end{aligned}$$

For $r \neq 1$,

$$a + ar + ar^2 + \cdots + ar^n = a\left(\frac{r^{n+1} - 1}{r - 1}\right).$$

Now $\phi + \phi^2 + \cdots + \phi^n$ is a geometric series with common ratio $\phi$, so

$$\begin{aligned}
\phi + \phi^2 + \cdots + \phi^n &= \phi(1 + \phi + \cdots + \phi^{n-1})\\
&= \phi\left(\frac{\phi^n - 1}{\phi - 1}\right)\\
&= \phi^2(\phi^n - 1),
\end{aligned}$$

because $\phi - 1 = 1/\phi$, by equation (5.3). A similar result holds for $\psi$:

$$\psi + \psi^2 + \cdots + \psi^n = \psi^2(\psi^n - 1).$$

So

$$\begin{aligned}
F_1 + F_2 + \cdots + F_n &= \frac{1}{\sqrt{5}}\left(\phi^2(\phi^n - 1) - \psi^2(\psi^n - 1)\right)\\
&= \frac{1}{\sqrt{5}}\left(\phi^{n+2} - \psi^{n+2}\right) - \frac{1}{\sqrt{5}}\left(\phi^2 - \psi^2\right)\\
&= F_{n+2} - F_2 = F_{n+2} - 1,
\end{aligned}$$

by Binet's formula, as required.

This second proof is in one sense more straightforward, since it does not rely on the good idea, but the algebraic details are more complicated. In the next activity, you are asked to prove a similar identity, using the first approach.

### *Activity 5.1 Proving another Fibonacci sum identity*

Prove the identity

$$F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}, \quad \text{for } n = 1, 2, 3, \ldots,$$

by rearranging the Fibonacci recurrence relation in the form:

$$F_{n+1} = F_{n+2} - F_n \quad (n = 0, 1, 2, \ldots),$$

and using telescoping cancellation.

A solution is given on page 41.

For example, for $n = 3$,

$$F_1 + F_3 + F_5 = 1 + 2 + 5,$$

and $F_6 = 8$.

## *5.2 Cassini's identity*

Next we discuss a property of Fibonacci numbers observed as long ago as 1680 by Giovanni Cassini. This concerns the expression $F_{n-1}F_{n+1} - F_n^2$, which we call $C_n$. The first few terms of this expression are given in Table 5.2.

Cassini (1625–1712) was an Italian astronomer, who used astronomical observations to map the Earth.

*Table 5.2* Cassini's expression

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $F_n$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 |
| $C_n = F_{n-1}F_{n+1} - F_n^2$ | | $-1$ | 1 | $-1$ | 1 | $-1$ | 1 |

See Exercise 2.2.
$C_n$ is not defined for $n = 0$.

On the basis of these values, it is natural to conjecture that $C_n$ always takes the value 1 when $n$ is even and the value $-1$ when $n$ is odd. This conjecture is true, as is shown below.

**Cassini's identity**

For $n = 1, 2, 3, \ldots,$

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$

Once again, there are several ways to prove this identity. For example, it is possible to use Binet's formula, but the algebraic details are quite tricky. Instead we use the 'good idea' of rearranging $C_{n+1} = F_nF_{n+2} - F_{n+1}^2$, using the Fibonacci recurrence relation twice.

You would certainly not be expected to think of this approach.

We have

$$\begin{aligned} C_{n+1} &= F_nF_{n+2} - F_{n+1}^2 \\ &= F_n(F_{n+1} + F_n) - F_{n+1}^2 \\ &= F_n^2 - F_{n+1}(F_{n+1} - F_n) \\ &= F_n^2 - F_{n+1}F_{n-1} \\ &= -C_n. \end{aligned}$$

Since this equation is true for $n = 1, 2, 3, \ldots$ and we know from Table 5.2 that $C_1 = -1$, we deduce that $C_2 = 1$, $C_3 = -1$, $C_4 = 1, \ldots$. Thus $C_n = (-1)^n$, for $n = 1, 2, 3, \ldots$, which is what we wanted to show.

In the next activity you are asked to *verify* a Cassini-type identity for a different linear second-order recurrence sequence.

### Activity 5.2 A Cassini-type identity

In this activity you are asked to show that the closed form you found in Activity 3.2(a),

$$u_n = 3^n + 4^n \quad (n = 0, 1, 2, \ldots),$$

satisfies the Cassini-type identity

$$u_{n-1}u_{n+1} - u_n^2 = 12^{n-1}, \quad \text{for } n = 1, 2, 3, \ldots.$$

(a) Use the appropriate rule for powers to simplify

$$3^{n-1} \times 3^{n+1}.$$

(b) Use the closed form for $u_n$ to obtain expressions for

$$u_{n-1}u_{n+1} \quad \text{and} \quad u_n^2.$$

(c) Hence show that

$$u_{n-1}u_{n+1} - u_n^2 = 12^{n-1}, \quad \text{for } n = 1, 2, 3, \ldots.$$

Solutions are given on page 42.

Cassini's identity is the basis of an amusing dissection paradox. Figure 5.1 shows a square and a rectangle, each apparently made from the same four pieces. Yet one has area 64 square units, whereas the other has area 65 square units. Explain!

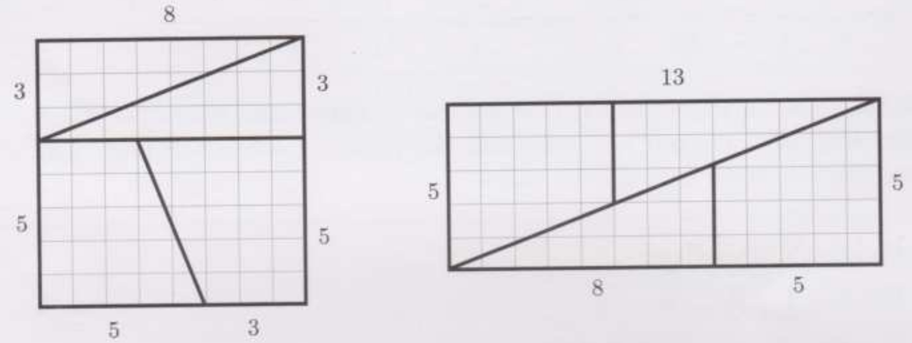


*Figure 5.1* A paradox

The solution of this paradox is as follows. When the four pieces from the left are assembled on the right, their outer edges form the rectangle but there is a space the shape of a slim parallelogram that remains uncovered in the middle. This parallelogram has area 1 square unit.

An accurate drawing will reveal this parallelogram.

Such a paradox can be created using any square whose side is a Fibonacci number with even subscript $n$, as shown in Figure 5.2.

In Figure 5.1, $n = 6$.

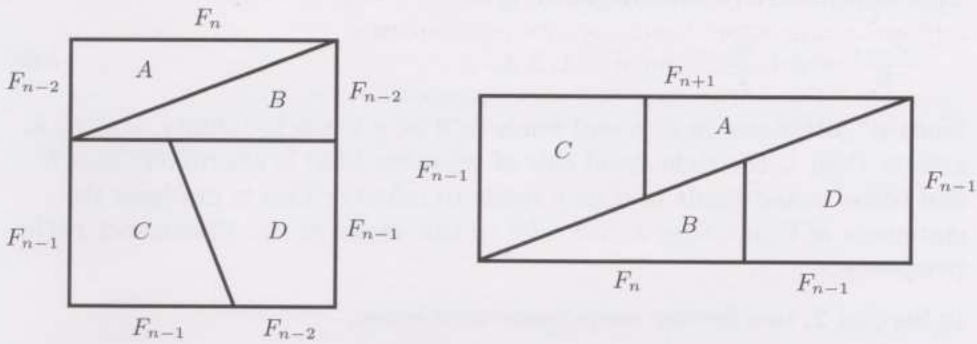


*Figure 5.2* Generalised paradox with $n$ even

The connection between the areas of the rectangle and the square arises from Cassini's identity which, when $n$ is even, gives

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n = 1.$$

## 5.3 Other conjectures

In this subsection, other conjectures that have arisen in this chapter are revisited briefly. As promised earlier, one of these conjectures is shown to be false.

This subsection will not be assessed.

In Section 2, we considered the ratios of successive pairs of Fibonacci numbers, given in the following table.

*Table 5.3* Fibonacci ratios (to three decimal places)

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_n$ | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 |
| $F_{n+1}/F_n$ | 1 | 2 | 1.5 | 1.667 | 1.6 | 1.625 | 1.615 | 1.619 | 1.618 | 1.618 | |

On the basis of these calculations, the following conjecture was made.

**Conjecture 2**
The terms of the sequence $F_{n+1}/F_n$ $(n = 1, 2, 3, \ldots)$ lie alternately above and below the golden ratio $\phi$, and

$$\frac{F_{n+1}}{F_n} \to \phi \text{ as } n \to \infty.$$

In Subsection 2.2 this conjecture was related to the appearance of Fibonacci numbers in nature.

This conjecture is true. There are various ways to prove it, but perhaps the neatest is to use the remarkable identity

$$F_{n+1} - \phi F_n = \psi^n, \quad \text{for } n = 0, 1, 2, \ldots,$$

which can be derived using Binet's formula as follows:

$$\begin{aligned} F_{n+1} - \phi F_n &= \frac{1}{\sqrt{5}}\left(\phi^{n+1} - \psi^{n+1}\right) - \frac{\phi}{\sqrt{5}}\left(\phi^n - \psi^n\right) \\ &= \frac{1}{\sqrt{5}}\left(-\psi^{n+1} + \phi\psi^n\right) \\ &= \psi^n\left(\frac{-\psi + \phi}{\sqrt{5}}\right) \\ &= \psi^n, \quad \text{by equation (5.2).} \end{aligned}$$

This identity can be rearranged to give

Division by $F_n$ is not possible for $n = 0$.

$$\frac{F_{n+1}}{F_n} = \phi + \frac{\psi^n}{F_n}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{5.6}$$

Since $\psi^n$ alternates in sign and tends to 0 as $n$ tends to infinity, and $F_n$ is greater than 1, the right-hand side of equation (5.6) is alternately above and below $\phi$ and tends to $\phi$ as $n$ tends to infinity. This is precisely the statement of Conjecture 2. We refer to this result as the **Fibonacci ratio property**.

In Section 2, two further conjectures were made.

**Conjecture 3**
If $n$ is a factor of $m$, then $F_n$ is a factor of $F_m$.

**Conjecture 4**
If $n$ is an odd prime number, then $F_n$ is a prime number.

Conjecture 3 was based on the evidence that

$F_3$ is a factor of $F_3, F_6, F_9, F_{12}, F_{15}$;

$F_4$ is a factor of $F_4, F_8, F_{12}, F_{16}$;

$F_5$ is a factor of $F_5, F_{10}, F_{15}$.

This conjecture is true and a proof can be given using Binet's formula. Moreover, the 'converse' result is also true, namely:

if $F_n$ is a factor of $F_m$, then $n$ is a factor of $m$.

Conjecture 3 and its converse tell us which Fibonacci numbers are divisible by any given Fibonacci number. For example, since $F_6 = 8$, the Fibonacci numbers that are divisible by 8 are precisely

$F_6, F_{12}, F_{18}, \ldots$.

Conjecture 4 was based on the evidence that

$F_3 = 2, F_5 = 5, F_7 = 13, F_{11} = 89, F_{13} = 233,$

are all prime numbers. This conjecture is false, however, and we do not have to search far for a counter-example. Although the next case $F_{17} = 1597$ is prime, the case after that, $F_{19} = 4181$, is not prime $(4181 = 37 \times 113)$.

Finally, it is worth noting that, in spite of the close scrutiny that Fibonacci numbers have received over the centuries, some problems about them remain unsolved. For example, at the time of writing it is not known whether there are infinitely many Fibonacci numbers which are prime numbers.

## Summary of Section 5

This section has introduced:

◇ the Fibonacci sum identity:
for $n = 1, 2, 3, \ldots$,

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1;$$

◇ Cassini's identity:
for $n = 1, 2, 3, \ldots$,

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n;$$

◇ the Fibonacci ratio property:
the terms of the sequence $F_{n+1}/F_n$ $(n = 1, 2, 3, \ldots)$ lie alternately above and below the golden ratio $\phi$, and

$$\frac{F_{n+1}}{F_n} \to \phi \text{ as } n \to \infty.$$

This section has also presented some examples of proofs, and the use of a counter-example to show that a conjecture is false.

## Exercises for Section 5

### Exercise 5.1

(a) Use the Fibonacci recurrence relation to prove the identity that, for $n = 1, 2, 3, \ldots$,

$$F_n^2 = F_nF_{n+1} - F_{n-1}F_n.$$

(b) Use part (a) and telescoping cancellation to deduce the identity that, for $n = 1, 2, 3, \ldots$,

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_nF_{n+1},$$

which was conjectured in Exercise 2.3.

### Exercise 5.2

(a) Show, by substitution, that

$$u_n = 3 \times 2^n + 2 \times (-3)^n \quad (n = 0, 1, 2, \ldots)$$

satisfies the recurrence system

$$u_0 = 5, u_1 = 0, \quad u_{n+2} = -u_{n+1} + 6u_n \quad (n = 0, 1, 2, \ldots).$$

(b) Show, by substitution, that the sequence $u_n$ in part (a) satisfies the Cassini-type identity

$$u_{n-1}u_{n+1} - u_n^2 = 150 \times (-6)^{n-1}, \quad \text{for } n = 1, 2, 3, \ldots.$$

# Summary of Chapter A1

In this chapter, you met two historical problems whose solutions led to various ways to explore mathematics: drawing diagrams, considering special cases, noticing patterns, forming conjectures, and using algebra to find solutions and prove identities.

## Learning outcomes

You have been working towards the following learning outcomes.

### Terms to know and use

The golden ratio, Fibonacci sequence, Fibonacci numbers, conjecture, counter-example, linear second-order recurrence system, linear second-order recurrence sequence, initial terms, general solution, Binet's formula, Binet's approximation, identity, telescoping cancellation, proof.

### Symbols to know and use

$\phi$, $\psi$, $F_n$

### Mathematical skills

- ◇ Explore special cases of a problem to get a feeling for it and the nature of its solution.
- ◇ Conjecture a general result from special cases.
- ◇ Apply general results in specific instances.
- ◇ Apply a strategy to obtain a closed form for a linear second-order recurrence system.
- ◇ Prove an identity by manipulating algebraic expressions and using known properties.

### Mathcad skills

- ◇ Create sequences using recurrence systems.

### Ideas to be aware of

- ◇ Using algebra to express geometrical relationships and solve geometrical problems.
- ◇ The relationship between a conjecture, a counter-example and a proof.

# Solutions to Activities

## Solution 1.1

(a) When the longer side is 1.6, the ratio for the smaller rectangle is

$$\frac{1}{0.6} = 1.667 \text{ (to 3 d.p.)},$$

which is greater than 1.6.

(b) When the longer side is 1.9, the ratio for the smaller rectangle is

$$\frac{1}{0.9} = 1.111 \text{ (to 3 d.p.)},$$

which is less than 1.9.

(c) When the longer side is 1.7, the ratio for the smaller rectangle is

$$\frac{1}{0.7} = 1.429 \text{ (to 3 d.p.)},$$

which is less than 1.7.

It appears that a solution to the Rectangle Problem is a ratio which lies between 1.6 and 1.7, probably nearer to 1.6.

## Solution 1.2

(a) For the quadratic equation $x^2 - x - 1 = 0$, where $a = 1$, $b = -1$ and $c = -1$, the formula gives

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-1)}}{2 \times 1} = \frac{1 \pm \sqrt{5}}{2}.$$

Thus the solutions are

$$x = \tfrac{1}{2}(1 + \sqrt{5}) = 1.618\ldots,$$

and

$$x = \tfrac{1}{2}(1 - \sqrt{5}) = -0.618\ldots.$$

(b) The negative value of $x$ is not a solution to the Rectangle Problem, since the length of each side of the rectangle must be positive. Therefore the only solution to the Rectangle Problem is

$$x = \tfrac{1}{2}(1 + \sqrt{5}) = 1.618\ldots.$$

## Solution 1.3

(a) The solution should look like this.

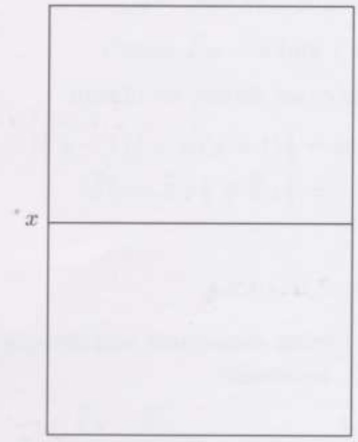


*Figure S.1*

(b) When the longer side is 1.5, the ratio for the smaller rectangle is

$$\frac{1}{0.75} = 1.333 \text{ (to 3 d.p.)},$$

which is less than 1.5. Therefore it seems likely that the solution to this rectangle problem is less than 1.5 (and greater than 1.333).

(c) If we take the shorter side to be 1 and denote the longer side by $x$, then the ratio of the lengths of the sides of each of the two smaller rectangles is

$$\frac{1}{x/2} = \frac{2}{x}.$$

We want this ratio to be the same as the ratio $x/1$ of the lengths of the sides of the original rectangle. Therefore $x$ must satisfy

$$\frac{x}{1} = \frac{2}{x}; \quad \text{that is,} \quad x^2 = 2.$$

The two solutions of this equation are $x = \pm\sqrt{2}$. Since the length of the side of a rectangle must be positive, the only solution of this rectangle problem is that the ratio of the lengths of the sides of the original rectangle is $x = \sqrt{2} = 1.414\ldots.$

## Solution 1.4

(a) Using the exact forms, we obtain

$$\begin{aligned}\phi\psi &= \tfrac{1}{2}(1+\sqrt{5}) \times \tfrac{1}{2}(1-\sqrt{5}) \\ &= \tfrac{1}{4}(1-5) = -1.\end{aligned}$$

(Did you apply the 'difference of two squares'

$$(a+b)(a-b) = a^2 - b^2,$$

with $a = 1$ and $b = \sqrt{5}$, here?)

(b) Using the exact forms, we obtain

$$\begin{aligned}\phi - \psi &= \tfrac{1}{2}(1+\sqrt{5}) - \tfrac{1}{2}(1-\sqrt{5}) \\ &= \tfrac{1}{2}\sqrt{5} + \tfrac{1}{2}\sqrt{5} = \sqrt{5}.\end{aligned}$$

## Solution 1.5

(a) On multiplying numerator and denominator by $\sqrt{5} - \sqrt{2}$, we obtain

$$\begin{aligned}\frac{1}{\sqrt{5}+\sqrt{2}} &= \frac{\sqrt{5}-\sqrt{2}}{(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})} \\ &= \frac{\sqrt{5}-\sqrt{2}}{5-2} \\ &= \tfrac{1}{3}(\sqrt{5}-\sqrt{2}).\end{aligned}$$

(b) On multiplying numerator and denominator by $2 + \sqrt{3}$, we obtain

$$\begin{aligned}\frac{2+\sqrt{3}}{2-\sqrt{3}} &= \frac{(2+\sqrt{3})(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})} \\ &= \frac{4+4\sqrt{3}+3}{4-3} \\ &= 7+4\sqrt{3}.\end{aligned}$$

## Solution 1.6

In each case, let the solutions be $\alpha$ and $\beta$.

(a) In this case,

$$\alpha+\beta = -\frac{-5}{1} = 5 \quad\text{and}\quad \alpha\beta = \frac{6}{1} = 6.$$

Therefore

$$\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{5}{6}.$$

(b) In this case,

$$\alpha+\beta = -\frac{1}{3} \quad\text{and}\quad \alpha\beta = \frac{-1}{3} = -\frac{1}{3}.$$

Therefore

$$\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{-1/3}{-1/3} = 1.$$

(c) In this case, since there is no term in $x$, $b = 0$. So

$$\alpha+\beta = -\frac{0}{5} = 0 \quad\text{and}\quad \alpha\beta = \frac{-2}{5} = -\frac{2}{5}.$$

Therefore

$$\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{0}{-2/5} = 0.$$

## Solution 1.7

Since $\phi$ satisfies the golden ratio equation, we have

$$\phi^2 - \phi - 1 = 0.$$

This can be rearranged in the form

$$\phi^2 = \phi + 1.$$

On multiplying throughout by $\phi$, we obtain

$$\phi^3 = \phi^2 + \phi.$$

On substituting $\phi^2 = \phi + 1$ into the right-hand side, we then obtain

$$\phi^3 = (\phi+1) + \phi = 2\phi + 1,$$

as required.

## Solution 2.1

$F_{15} = 377 + 233 = 610$ and $F_{16} = 610 + 377 = 987$.

## Solution 2.2

(a) The sums are:

$$\begin{aligned}&F_1 = 1,\\ &F_1 + F_2 = 1 + 1 = 2,\\ &F_1 + F_2 + F_3 = 1 + 1 + 2 = 4,\\ &F_1 + F_2 + \cdots + F_4 = 4 + 3 = 7,\\ &F_1 + F_2 + \cdots + F_5 = 7 + 5 = 12,\\ &F_1 + F_2 + \cdots + F_6 = 12 + 8 = 20,\\ &F_1 + F_2 + \cdots + F_7 = 20 + 13 = 33,\\ &F_1 + F_2 + \cdots + F_8 = 33 + 21 = 54.\end{aligned}$$

(b) These sums are all one less than a Fibonacci number. In fact, they are one less than the Fibonacci number which is two places further along the Fibonacci sequence. For example,

$$F_1 + F_2 + \cdots + F_8 = 54 = F_{10} - 1.$$

This suggests that, in general, the sum $F_1 + F_2 + \cdots + F_n$ is equal to $F_{n+2} - 1$.

## Solution 2.3

(a) The ratios are (to three decimal places):

$$F_2/F_1 = 1/1 = 1,$$

$$F_3/F_2 = 2/1 = 2,$$

$$F_4/F_3 = 3/2 = 1.5,$$

$$F_5/F_4 = 5/3 = 1.667,$$

$$F_6/F_5 = 8/5 = 1.6,$$

$$F_7/F_6 = 13/8 = 1.625,$$

$$F_8/F_7 = 21/13 = 1.615,$$

$$F_9/F_8 = 34/21 = 1.619.$$

(b) This sequence of ratios appears to tend to a value, which is close to $\phi = 1.618\ldots$, and the terms appear to lie alternately above and below that value. This suggests that, in general, the terms of the sequence

$$F_{n+1}/F_n \quad (n = 1, 2, 3, \ldots)$$

lie alternately above and below $\phi$, and tend to $\phi$ in the long term; that is,

$$\frac{F_{n+1}}{F_n} \to \phi \text{ as } n \to \infty.$$

## Solution 3.2

(a) The auxiliary equation is

$$r^2 - 7r + 12 = (r-3)(r-4) = 0.$$

This has solutions $r = 3$ and $r = 4$.

(Alternatively, these solutions could be found by using the quadratic equation formula.)

Thus the general solution is

$$u_n = A3^n + B4^n,$$

where $A$ and $B$ are unknown constants.

To find $A$ and $B$, we use the initial terms:

$$u_0 = 2 \quad \text{gives} \quad A + B = 2,$$

$$u_1 = 7 \quad \text{gives} \quad 3A + 4B = 7.$$

Hence $A = 1$ and $B = 1$, so

$$u_n = 3^n + 4^n \quad (n = 0, 1, 2, \ldots).$$

(b) The auxiliary equation is

$$r^2 - 6r + 9 = (r-3)^2 = 0.$$

This has the (repeated) solution $r = 3$.

Thus the general solution is

$$u_n = (A + Bn)3^n,$$

where $A$ and $B$ are unknown constants.

To find $A$ and $B$, we use the initial terms:

$$u_0 = 2 \quad \text{gives} \quad A = 2,$$

$$u_1 = 7 \quad \text{gives} \quad (A + B)3 = 7.$$

Hence $A = 2$ and $B = 1/3$, so

$$u_n = (2 + \tfrac{1}{3}n)3^n \quad (n = 0, 1, 2, \ldots).$$

(c) The auxiliary equation is

$$r^2 - 2r - 8 = (r-4)(r+2) = 0.$$

This has solutions $r = 4$ and $r = -2$.

Thus the general solution is

$$u_n = A4^n + B(-2)^n,$$

where $A$ and $B$ are unknown constants.

To find $A$ and $B$, we use the initial terms:

$$u_0 = 2 \quad \text{gives} \quad A + B = 2,$$

$$u_1 = 7 \quad \text{gives} \quad 4A - 2B = 7.$$

Hence $A = 11/6$, $B = 1/6$, so

$$u_n = \frac{11}{6}4^n + \frac{1}{6}(-2)^n \quad (n = 0, 1, 2, \ldots).$$

## Solution 3.3

With $n = 20$, a calculator gives

$$\frac{\phi^{20}}{\sqrt{5}} = 6765.0000.$$

Thus, by Binet's approximation, $F_{20} = 6765$.

## Solution 5.1

We deduce from

$$F_{n+1} = F_{n+2} - F_n \quad (n = 0, 1, 2, \ldots)$$

that the following $n$ equations are true:

$$\begin{aligned} F_1 &= F_2 - F_0 \\ F_3 &= F_4 - F_2 \\ F_5 &= F_6 - F_4 \\ &\cdots \\ F_{2n-1} &= F_{2n} - F_{2n-2}. \end{aligned}$$

If we add these equations, then

- ◇ on the left-hand side, we obtain the sum $F_1 + F_3 + \cdots + F_{2n-1}$;
- ◇ on the right-hand side, we obtain $F_{2n} - F_0$, by telescoping cancellation.

Since $F_0 = 0$,

$$F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}, \quad \text{for } n = 1, 2, 3, \ldots,$$

as required.

### *Solution 5.2*

(a) Using the rule $a^p a^q = a^{p+q}$, we have

$$3^{n-1} \times 3^{n+1} = 3^{(n-1)+(n+1)} = 3^{2n}.$$

(b) Using the closed form $u_n = 3^n + 4^n$, we obtain

$$\begin{aligned} u_{n-1}u_{n+1} &= (3^{n-1} + 4^{n-1})(3^{n+1} + 4^{n+1}) \\ &= 3^{2n} + 3^{n-1}4^{n+1} + 4^{n-1}3^{n+1} + 4^{2n}, \end{aligned}$$

and

$$\begin{aligned} u_n^2 &= (3^n + 4^n)^2 \\ &= 3^{2n} + 2(3^n 4^n) + 4^{2n}. \end{aligned}$$

(c) Therefore, after cancelling, we obtain

$$\begin{aligned} u_{n-1}u_{n+1} - u_n^2 &= 3^{n-1}4^{n+1} + 4^{n-1}3^{n+1} \\ &\quad - 2(3^n 4^n) \\ &= 3^{n-1}4^{n-1}(4^2 + 3^2 - 2 \times 3 \times 4) \\ &= 12^{n-1}, \end{aligned}$$

as required.

# Solutions to Exercises

## Solution 1.1

The following diagram illustrates the problem.

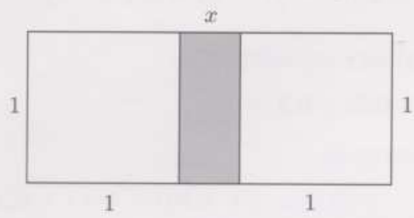


*Figure S.2*

The original rectangle has sides 1 and $x$ (where $x > 2$), so the shaded rectangle has sides $x - 2$ and 1. For these rectangles to be similar, we need

$$\frac{x}{1} = \frac{1}{x-2},$$

which simplifies to give

$$x(x-2) = 1; \quad \text{that is,} \quad x^2 - 2x - 1 = 0.$$

The solutions of this quadratic equation are

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(-1)}}{2} \\ &= \tfrac{1}{2}(2 \pm \sqrt{8}) = 1 \pm \sqrt{2}, \end{aligned}$$

because $\sqrt{8} = 2\sqrt{2}$. Since $\sqrt{2} = 1.414\ldots$ and the solution of the problem must be positive, the required ratio of lengths is $x = 1 + \sqrt{2} = 2.414$ (to 3 d.p.).

(The fact that this number is greater than 2 provides a check on the answer.)

## Solution 1.2

(a) In this case,

$$\alpha + \beta = -\frac{-4}{1} = 4 \quad \text{and} \quad \alpha\beta = \frac{3}{1} = 3.$$

Using the given identity, we obtain

$$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = 16 - 6 = 10.$$

(b) In this case,

$$\alpha + \beta = -\frac{-2}{4} = \tfrac{1}{2} \quad \text{and} \quad \alpha\beta = \frac{-1}{4} = -\tfrac{1}{4}.$$

Therefore

$$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \tfrac{1}{4} - 2(-\tfrac{1}{4}) = \tfrac{3}{4}.$$

(c) In this case,

$$\alpha + \beta = 0 \quad \text{and} \quad \alpha\beta = \frac{-2}{3} = -\tfrac{2}{3}.$$

Therefore

$$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = 0 - 2(-\tfrac{2}{3}) = \tfrac{4}{3}.$$

## Solution 1.3

(a) For $n = 0$,

$$\phi^0 + \psi^0 = 1 + 1 = 2.$$

For $n = 1$,

$$\phi^1 + \psi^1 = \phi + \psi = 1.$$

By the golden ratio equation,

$$\phi^2 = \phi + 1 \quad \text{and} \quad \psi^2 = \psi + 1,$$

so

$$\begin{aligned} \phi^2 + \psi^2 &= (\phi+1) + (\psi+1) \\ &= \phi + \psi + 2 = 3, \end{aligned}$$

since $\phi + \psi = 1$.

Multiplying $\phi^2 = \phi + 1$ by $\phi$ and $\psi^2 = \psi + 1$ by $\psi$, we obtain

$$\phi^3 = \phi^2 + \phi \quad \text{and} \quad \psi^3 = \psi^2 + \psi.$$

Therefore

$$\begin{aligned} \phi^3 + \psi^3 &= (\phi^2 + \phi) + (\psi^2 + \psi) \\ &= \phi^2 + \psi^2 + \phi + \psi \\ &= 3 + 1 = 4, \end{aligned}$$

since $\phi^2 + \psi^2 = 3$ and $\phi + \psi = 1$.

Repeating this process, we obtain

$$\phi^4 = \phi^3 + \phi^2 \quad \text{and} \quad \psi^4 = \psi^3 + \psi^2,$$

so

$$\begin{aligned} \phi^4 + \psi^4 &= (\phi^3 + \phi^2) + (\psi^3 + \psi^2) \\ &= \phi^3 + \psi^3 + \phi^2 + \psi^2 \\ &= 4 + 3 = 7, \end{aligned}$$

since $\phi^3 + \psi^3 = 4$ and $\phi^2 + \psi^2 = 3$.

(b) The answers are $2, 1, 3, 4, 7$. One pattern which seems to be emerging is that each answer (after the second) is the sum of the previous two answers:

$$3 = 1 + 2,\ 4 = 3 + 1,\ 7 = 4 + 3.$$

## Solution 2.1

We calculate $F_{n+1}/F_n$, for $n = 9$ and $n = 10$:

$$\frac{F_{10}}{F_9} = \frac{55}{34} = 1.617\,647\,1 \text{ (to 8 s.f.)}$$

and

$$\frac{F_{11}}{F_{10}} = \frac{89}{55} = 1.6\dot{1}\dot{8}.$$

These values continue the pattern of lying alternately above and below $\phi = 1.618\,033\,988\,7$, and they are

closer to $\phi$ than the earlier Fibonacci ratios. So they do *not* provide a counter-example to Conjecture 2.

## Solution 2.2

(a) The values of the expression $F_{n-1}F_{n+1} - F_n^2$, for $n = 1, 2, \ldots, 6$, are

$$F_0F_2 - F_1^2 = 0 \times 1 - 1^2 = -1,$$
$$F_1F_3 - F_2^2 = 1 \times 2 - 1^2 = 1,$$
$$F_2F_4 - F_3^2 = 1 \times 3 - 2^2 = -1,$$
$$F_3F_5 - F_4^2 = 2 \times 5 - 3^2 = 1,$$
$$F_4F_6 - F_5^2 = 3 \times 8 - 5^2 = -1,$$
$$F_5F_7 - F_6^2 = 5 \times 13 - 8^2 = 1.$$

(b) The expression appears to take the value 1 when $n$ is even and $-1$ when $n$ is odd. This suggests the conjecture that, for $n = 1, 2, 3, \ldots$,

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$

## Solution 2.3

(a) The values of the expression $F_1^2 + F_2^2 + \cdots + F_n^2$, for $n = 1, 2, \ldots, 6$, are

$$F_1^2 = 1^2 = 1,$$
$$F_1^2 + F_2^2 = 1^2 + 1^2 = 2,$$
$$F_1^2 + F_2^2 + F_3^2 = 1^2 + 1^2 + 2^2 = 6,$$
$$F_1^2 + F_2^2 + \cdots + F_4^2 = 6 + 3^2 = 15,$$
$$F_1^2 + F_2^2 + \cdots + F_5^2 = 15 + 5^2 = 40,$$
$$F_1^2 + F_2^2 + \cdots + F_6^2 = 40 + 8^2 = 104.$$

(b) It is harder to spot a pattern in this case. Notice, however, that

$$6 = 2 \times 3,\ 15 = 3 \times 5,\ 40 = 5 \times 8,$$

are all products of successive Fibonacci numbers, as are $1, 2$ and $104$. This suggests the conjecture that, for $n = 1, 2, 3, \ldots$,

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_nF_{n+1}.$$

## Solution 3.1

(a) The auxiliary equation is

$$r^2 - \tfrac{3}{2}r - 1 = 0.$$

By the formula,

$$r = \frac{-(-3/2) \pm \sqrt{(-3/2)^2 - 4 \times 1 \times (-1)}}{2 \times 1}$$
$$= \frac{3/2 \pm \sqrt{25/4}}{2}$$
$$= \frac{3/2 \pm 5/2}{2},$$

so the solutions are $r = 2$ and $r = -\frac{1}{2}$.

Thus the general solution is

$$u_n = A2^n + B\left(-\tfrac{1}{2}\right)^n,$$

where $A$ and $B$ are unknown constants.

To find $A$ and $B$, we use the initial terms:

$$u_0 = 0 \quad \text{gives} \quad A + B = 0,$$

$$u_1 = \tfrac{5}{2} \quad \text{gives} \quad 2A - \tfrac{1}{2}B = \tfrac{5}{2}.$$

Hence $A = 1$ and $B = -1$, so

$$u_n = 2^n - (-\tfrac{1}{2})^n \quad (n = 0, 1, 2, \ldots).$$

(b) The auxiliary equation is

$$r^2 - 0.9r + 0.2 = 0.$$

By the formula,

$$r = \frac{-(-0.9) \pm \sqrt{(-0.9)^2 - 4 \times 1 \times 0.2}}{2 \times 1}$$
$$= \frac{0.9 \pm \sqrt{0.01}}{2}$$
$$= \frac{0.9 \pm 0.1}{2},$$

so the solutions are $r = 0.5$ and $r = 0.4$.

Thus the general solution is

$$u_n = A(0.5)^n + B(0.4)^n,$$

where $A$ and $B$ are unknown constants.

To find $A$ and $B$, we use the initial terms:

$$u_0 = 2 \quad \text{gives} \quad A + B = 2,$$

$$u_1 = 7 \quad \text{gives} \quad 0.5A + 0.4B = 7.$$

Hence $A = 62$ and $B = -60$, so

$$u_n = 62(0.5)^n - 60(0.4)^n \quad (n = 0, 1, 2, \ldots).$$

(c) The auxiliary equation is

$$r^2 + r + \tfrac{1}{4} = 0.$$

By the formula,

$$r = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times \frac{1}{4}}}{2 \times 1}$$
$$= \frac{-1}{2}.$$

So the (repeated) solution is $r = -\frac{1}{2}$.

Thus the general solution is

$$u_n = (A + Bn)(-\tfrac{1}{2})^n,$$

where $A$ and $B$ are unknown constants.

To find $A$ and $B$, we use the initial terms:

$$u_0 = -1 \quad \text{gives} \quad A = -1,$$

$$u_1 = 2 \quad \text{gives} \quad (A + B)(-\tfrac{1}{2}) = 2.$$

Hence $A = -1$ and $B = -3$, so

$$u_n = (-1 - 3n)(-\tfrac{1}{2})^n \quad (n = 0, 1, 2, \ldots).$$

## Solution 3.2

(a) Applying the recurrence relation repeatedly, we obtain the following table of values.

*Table S.1* Lucas numbers

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $L_n$ | 2 | 1 | 3 | 4 | 7 | 11 | 18 |

(b) The auxiliary equation is

$$r^2 - r - 1 = 0,$$

which has solutions $r = \phi = 1.618\ldots$ and $r = \psi = -0.618\ldots$.

Thus the general solution is

$$u_n = A\phi^n + B\psi^n,$$

where $A$ and $B$ are unknown constants.

We use the initial terms to find $A$ and $B$ for the Lucas sequence:

$$L_0 = 2 \quad \text{gives} \quad A + B = 2,$$

$$L_1 = 1 \quad \text{gives} \quad A\phi + B\psi = 1.$$

Eliminating $B$, we obtain $A(\phi - \psi) = 1 - 2\psi$. Since

$$\phi - \psi = \sqrt{5} \quad \text{(Activity 1.4(b))}$$

and

$$1 - 2\psi = 1 - 2 \times \tfrac{1}{2}(1 - \sqrt{5}) = \sqrt{5},$$

we obtain $A = 1$ and $B = 1$. (These values for the constants are easily checked, since we know that $\phi + \psi = 1$.) Hence

$$L_n = \phi^n + \psi^n \quad (n = 0, 1, 2, \ldots).$$

(You may remember calculating the first few terms of this sequence in Exercise 1.3.)

(c) The above closed form gives

$$L_{15} = \phi^{15} + \psi^{15}.$$

Since $L_n$ is an integer and $-\frac{1}{2} < \psi^{15} < \frac{1}{2}$, $L_{15}$ is the nearest integer to $\phi^{15}$. Now

$$\phi^{15} = \left(\tfrac{1}{2}(1 + \sqrt{5})\right)^{15} = 1364.000\,7,$$

so $L_{15} = 1364$.

## Solution 5.1

(a) The Fibonacci recurrence relation in the form $F_{n+1} = F_n + F_{n-1}$ $(n = 1, 2, 3, \ldots)$ can be rearranged as $F_n = F_{n+1} - F_{n-1}$. On multiplying throughout by $F_n$, we obtain, for $n = 1, 2, 3, \ldots$,

$$F_n^2 = F_n F_{n+1} - F_{n-1} F_n,$$

as required.

(b) The identity in part (a) gives the $n$ equations

$$\begin{aligned} F_1^2 &= F_1 F_2 - F_0 F_1, \\ F_2^2 &= F_2 F_3 - F_1 F_2, \\ F_3^2 &= F_3 F_4 - F_2 F_3, \\ &\ldots \\ F_n^2 &= F_n F_{n+1} - F_{n-1} F_n. \end{aligned}$$

If we add these equations, then

◇ on the left-hand side, we obtain the sum $F_1^2 + F_2^2 + \cdots + F_n^2$;

◇ on the right-hand side, we obtain $F_n F_{n+1} - F_0 F_1$, by telescoping cancellation.

Now $F_0 = 0$, so for $n = 1, 2, 3, \ldots$,

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1},$$

as required.

## Solution 5.2

(a) First we check the initial terms. We have

$$\begin{aligned} u_0 &= 3 \times 2^0 + 2 \times (-3)^0 = 3 + 2 = 5, \\ u_1 &= 3 \times 2^1 + 2 \times (-3)^1 = 6 - 6 = 0, \end{aligned}$$

as required.

To show that the recurrence relation is satisfied we substitute for $u_{n+2}$ in the left-hand side (LHS) and for $u_{n+1}$ and $u_n$ in the right-hand side (RHS), and show that the two sides are equal. We have, for $n = 0, 1, 2, \ldots$,

$$\begin{aligned} \text{LHS } &= u_{n+2} = 3 \times 2^{n+2} + 2 \times (-3)^{n+2} \\ &= 12 \times 2^n + 18 \times (-3)^n; \end{aligned}$$

$$\begin{aligned} \text{RHS } &= -u_{n+1} + 6u_n \\ &= -\left(3 \times 2^{n+1} + 2 \times (-3)^{n+1}\right) \\ &\quad + 6\left(3 \times 2^n + 2 \times (-3)^n\right) \\ &= (-6 + 18)2^n + (6 + 12)(-3)^n \\ &= 12 \times 2^n + 18 \times (-3)^n. \end{aligned}$$

So LHS = RHS.

(b) We substitute for $u_n$, $u_{n-1}$ and $u_{n+1}$ in the LHS, $u_{n-1}u_{n+1} - u_n^2$, and show that the result simplifies to $150 \times (-6)^{n-1}$, which is the RHS. We have, for $n = 1, 2, 3, \ldots$,

$$\begin{aligned} u_{n-1}u_{n+1} &= (3 \times 2^{n-1} + 2 \times (-3)^{n-1}) \\ &\quad \times (3 \times 2^{n+1} + 2 \times (-3)^{n+1}) \\ &= 9 \times 2^{2n} + 6 \times 2^{n-1} \times (-3)^{n+1} \\ &\quad + 6 \times (-3)^{n-1} \times 2^{n+1} + 4 \times (-3)^{2n}, \end{aligned}$$

and

$$\begin{aligned} u_n^2 &= (3 \times 2^n + 2 \times (-3)^n)^2 \\ &= 9 \times 2^{2n} + 12 \times 2^n \times (-3)^n + 4 \times (-3)^{2n}. \end{aligned}$$

Therefore, after cancelling, we obtain

$$\begin{aligned} \text{LHS} &= 2^{n-1} \times (-3)^{n-1} \\ &\quad \times (6 \times (-3)^2 + 6 \times 2^2 - 12 \times 2 \times (-3)) \\ &= 150 \times (-6)^{n-1} = \text{RHS, as required.} \end{aligned}$$

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